Indefinite Sturm-Liouville operators $(\operatorname{sgn} x)(-\frac{d^2}{dx^2} + q(x))$ with finite-zone potentials

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Abstract

The indefinite Sturm-Liouville operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ is studied. It is proved that similarity of A to a selfadjoint operator is equivalent to integral estimates of Cauchy integrals. Also similarity conditions in terms of Weyl functions are given. For operators with a finite-zone potential, the components A_{ess} and A_{disc} of A corresponding to essential and discrete spectrums, respectively, are considered. A criterion of similarity of A_{ess} to a selfadjoint operator is given in terms of Weyl functions for the Sturm-Liouville operator $-d^2/dx^2 + q(x)$ with a finite-zone potential q. Jordan structure of the operator A_{disc} is described. We present an example of the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ such that A is nondefinitizable and A is similar to a normal operator.

Keywords: J-selfadjoint operator, indefinite weight, nonselfadjoint operator, Sturm-Liouville operator, eigenvalue, algebraic multiplicity, geometric multiplicity, similarity, weighted norm inequalities.

1 Introduction

The main object of the paper is a nonselfadjoint indefinite Sturm-Liouville operator

$$A = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + q(x)\right) =: JL, \qquad \operatorname{dom} A = \operatorname{dom} L, \tag{1.1}$$

where $J : \ f \to \operatorname{sgn} x \cdot f(\cdot)$ and $L := -\frac{d^2}{dx^2} + q(x)$ is a selfadjoint Sturm-Liouville operator on $L^2(\mathbb{R})$ with a real continuous potential $q(\cdot)$. Differential operators with indefinite weights have intensively been investigated during two last decades (see [26, 4, 7, 52, 56, 8, 16, 58, 17, 28, 15, 50, 33]). The operator (1.1) on a finite interval subject to selfadjoint boundary conditions has discrete spectrum. The Riesz basis property of Dirichlet and other boundary value problem for Sturm-Liouville operators with indefinite weights has been investigated in [26, 4, 7, 52, 56, 50].

In general, the operator (1.1) considered on $L^2(\mathbb{R})$ has continuous spectrum. In this case in place of the Riesz basis property one considers the property of similarity either to a normal or to a selfadjoint operator.

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Let us recall that two closed operators T_1 and T_2 in a Hilbert space $\mathfrak{H}$ are called similar if there exist bounded operator V with the bounded inverse V^{-1} in $\mathfrak{H}$ such that $V\,\mathrm{dom}(T_1) = \mathrm{dom}(T_2)$ and $T_2 = VT_1V^{-1}$.

Using the Krein-Langer technique of definitizable operators in Krein spaces Ćurgus and Langer [7] have obtained the first result in this direction. In particular, their result yields that the J-selfadjoint operator (1.1) is similar to a selfadjoint operator if L is a uniformly positive operator (i.e., $L \geq \delta > 0$). Similarity of the operator $(\operatorname{sgn} x)\frac{d^2}{dx^2}$ to selfadjoint one was proved by Ćurgus and Najman [8]. Later on, one of the authors [30, 28] reproved this result using another approach. More precisely, using the resolvent criterion of similarity to a selfadjoint operator [45, 40] (see also Theorem 3.12 below) he proved in [30, 28] that the operator $A = (\operatorname{sgn} x) \cdot p(-i\frac{d}{dx})$ is similar to a selfadjoint operator if and only if the polynomial p is nonnegative.

Further, Faddeev and Shterenberg [15] investigated operator (1.1) with decaying potential. They shown, that A is similar to a selfadjoint operator if $L \geq 0$ and $\int_{\mathbb{R}}(1 + x^2)|q(x)|dx < \infty$.

The paper under consideration consists of two parts. In the first part we investigate the operator A assuming only that $q(\cdot)$ is continuous. We investigate this operator in the framework of extension theory considering it as a (nonselfadjoint) extension of the minimal symmetric operator

$$A_{\min} = A^+_{\min} \oplus A^-_{\min} = L^+_{\min} \oplus (-L^-_{\min}),$$

where $L^+_{\min}$ and $L^-_{\min}$ are minimal Sturm-Liouville operators generated by the differential expression L in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively. Here $\mathrm{dom}\, L^{\pm}_{\min} := \{f \in \mathrm{dom}\, L : \ P_{\pm}f \in \mathrm{dom}\, L\}$, where $P_{\pm}$ is the orthoprojection in $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_{\pm})$.

With operators $L^{\pm}_{\min}$ one associates the Weyl functions $m_{\pm}(\lambda)$ corresponding to the extensions $L^{\pm}_N$ of $L^{\pm}_{\min}$, $\mathrm{dom}\, L^{\pm}_N = \{f \in \mathrm{dom}\, L : \ f'(\pm 0) = 0\}$.

We obtain necessary and sufficient conditions of similarity in terms of the Weyl functions $M_+(\lambda) := m_+(\lambda)$ and $M_-(\lambda) = -m_-(\lambda)$. Note, that $M_{\pm}$ are R-functions (Nevanlinna-Herglotz functions), hence the limit values $M_{\pm}(x) := M_{\pm}(x + i0)$ exist a.e. on $\mathbb{R}$.

It is worth to note that the similarity problem for the operator A gives rise to two weight estimates for the Hilbert transform in $L^2(\mathbb{R})$. If fact, we show that the following estimate

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_{\pm}(t) + \operatorname{Im} M_{\mp}(t)}{|M_+(t) - M_-(t)|^2} \left|g^{\pm}(t)\Sigma'_{ac\pm}(t) + (H(g^{\pm} \cdot d\Sigma_{\pm})(t)\right|^2 dt \leq K_1 \int_{\mathbb{R}} |g^{\pm}(t)|^2 d\Sigma_{\pm}(t), \quad (1.2)$$

(see Theorem 5.2) gives two necessary conditions for the operator A to be similar to a selfadjoint operator.

We conjecture that, under the assumption $\sigma_{disc}(A) = \emptyset$, these estimates are also sufficient for similarity to a selfadjoint operator.

We show that the condition (1.2) yields the following necessary condition for similarity

$$\left(\frac{1}{|\mathcal{I} \cap E_{\pm}|} \int_{\mathcal{I}} \frac{\operatorname{Im} M_{\pm}(t)}{|M_+(t) - M_-(t)|^2} dt\right) \cdot \left(\frac{1}{|\mathcal{I} \cap E_{\pm}|} \int_{\mathcal{I}} \operatorname{Im} M_{\pm}(t) dt\right) < C, \quad (1.3)$$

where $\mathcal{I}(\subset \mathbb{R})$ is any interval, $E_{\pm}$ stand for the topological supports of $\operatorname{Im} M_{\pm}$, $E_{\pm} = \operatorname{supp}(M_{\pm})$, and C does not depend on $\mathcal{I}$.

In turn, (1.3) implies the following weaker (and simpler) necessary condition of similarity

$$\frac{\operatorname{Im} M_+(t) + \operatorname{Im} M_-(t)}{M_+(t) - M_-(t)} \in L^{\infty}(\mathbb{R}). \quad (1.4)$$

Moreover, we show that the stronger condition

$$\sup_{\lambda\in\mathbb{C}_+} \frac{|M_+(\lambda)+M_-(\lambda)|}{|M_+(\lambda)-M_-(\lambda)|} \ < \ \infty \tag{1.5}$$

is sufficient for similarity to a selfadjoint operator.

The second part of the paper is devoted to the spectral analysis of the operator (1.1) with a finite-zone potential $q(\cdot)$.

Recall that a quasiperiodic (in particular, periodic) potential $q(\cdot) = \overline{q(\cdot)}$ is called a finite-zone potential if the spectrum $\sigma(L)$ of the operator L has a finite number of bands (equivalently, the resolvent set $\rho(L)$ has a finite number of gaps = forbidden zones).

We show that the operator $A = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2}+q(x)\right)$ with a finite-zone potential q has a finite number of complex eigenvalues, and A has no (embedding) eigenvalues on the essential spectrum $\sigma_{ess}(A)$, that is $\sigma_p(A)\cap\sigma_{ess}(A)=\emptyset$ (equivalently, the essential spectrum of A coincides with purely continuous spectrum). Moreover, we show that the operator A admits the following direct sum decomposition:

$$A = A_{disc} \dotplus A_{ess},$$

where A_{ess} is a part of the operator A corresponding to essential spectrum $\sigma_{ess}(A)$ of A.

We summarize our main results (Theorem 7.2 and Corollary 7.4) as follows:

If the potential q is finite-zone, then the part A_{ess} of the operator A is similar to a selfadjoint operator if and only if condition (1.4) *is satisfied. Moreover, in this case A_{ess} is similar to a selfadjoint operator with absolutely continuous spectrum.*

The main results of the paper have been announced in our short communication [33].

Notations.

Throughout the paper we use the following notation. Let T be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows $\operatorname{dom}(T)$, $\ker(T)$, $\operatorname{ran}(T)$ are the domain, kernel, range of T, respectively. By $\operatorname{Lat} T$ we denote the set of invariant subspaces of a linear operator T. $\operatorname{span}\{f_1, f_2, \dots\}$ is the closed linear hull of vectors $f_1,\ f_2,\ \dots$. We denote by $\sigma(T)$ the spectrum of T. The discrete spectrum $\sigma_{disc}(T)$ is the set of isolated eigenvalues of finite algebraic multiplicity; the essential spectrum is defined by $\sigma_{ess}(T) := \sigma(T)\setminus\sigma_{disc}(T)$; $\sigma_p(T)$ stands for the set of eigenvalues; $\rho(T)$ is the resolvent set of T; $R_T(\lambda)$ is the resolvent of T,

$$R_T(\lambda) := (T-\lambda I)^{-1}, \quad \lambda\in\rho(T).$$

The continuous spectrum is defined by

$$\sigma_c(T) := \{\lambda\in\mathbb{C}\setminus\sigma_p(T) : \operatorname{ran}(T-\lambda) \neq \overline{\operatorname{ran}(T-\lambda)} = \mathfrak{H}\ \}.$$

Let $\sigma_{ac}(T)$ and $\sigma_s(T)$ denote the absolutely continuous and singular spectra of a selfadjoint operator T (see, for example, [1]).

Let $\mathcal{I}$ be an interval in $\mathbb{R}$. Let $d\Sigma$ be a Borel measure on $\mathcal{I}$. $L^2(\mathcal{I}, d\Sigma)$ is the Hilbert space of measurable functions f on $\mathcal{I}$ which satisfy $\int_{\mathcal{I}} |f|^2 d\Sigma < \infty$. If $\mathcal{I}$ or $d\Sigma$ is fixed, we will write $L^2(d\Sigma)$ or $L^2(\mathcal{I})$. The topological support $\operatorname{supp} d\Sigma$ of $d\Sigma$ is the smallest closed set S such that $d\Sigma(\mathbb{R}\setminus S)=0$. We denote the indicator function of a set S by $\chi_S(\cdot)$; $\chi_\pm(t) := \chi_{\mathbb{R}_\pm}(t)$.

We say $f\in H(\mathcal{D})$ if $f(\cdot)$ is a holomorphic function on a domain $\mathcal{D}$. By $\mathcal{N}^+(\mathbb{C}_+)$ we denote the Smirnov class on $\mathbb{C}_+$ (see Subsection 2.6). Suppose $\mathcal{I}$ be an interval in $\mathbb{R}$; then by $\operatorname{Lip}^\alpha(\mathcal{I})$, $\alpha\in(0,1]$, we denote the Lipschitz classes on $\mathcal{I}$ (see, for example, [18]).

We write $f(x) \asymp g(x) \quad (x\to x_0)$, if the functions $\frac{f}{g}$ and $\frac{g}{f}$ are bounded in a sufficiently small neighborhood of the point x_0; $f(x)\asymp g(x) \quad (x\in D)$ means that $\frac{f}{g}$ and $\frac{g}{f}$ are bounded on the set D.

2 Preliminaries

2.1 Indefinite Sturm-Liouville operators $(\operatorname{sgn} x)(-\frac{d^2}{dx^2} + q(x))$

Denote by J the multiplication operator by $\operatorname{sgn} x$ in the Hilbert space $L^2(\mathbb{R})$, $J: \ f(x) \to \operatorname{sgn} x f(x)$. Next we consider in $L^2(\mathbb{R})$ the differential expression

$$L = -\frac{d^2}{dx^2} + q(x) \ , \tag{2.1}$$

with a real continuous potential q. Suppose additionally that the minimal operators L^+_{min}, L^-_{min} (see [46], [47]) associated with (2.1) in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, have the deficiency indices $(1,1)$. Denote also by L the Sturm-Liouville operator generated in $L^2(\mathbb{R})$ by the differential expression (2.1). It is clear that L is selfadjoint in $L^2(\mathbb{R})$.

The main object of our paper is an indefinite Sturm-Liouville operator

$$A := JL = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + q(x)\right) \ , \qquad \operatorname{dom}(A) := \operatorname{dom}(L), \tag{2.2}$$

in $L^2(\mathbb{R})$. It is easy to see that $A \neq A^*$. Indeed, the operator $A^* = LJ$ is defined by the same differential expression (2.2) on the domain, $\operatorname{dom}(A^*) = J \operatorname{dom} L \neq \operatorname{dom}(A)$, containing functions discontinuous at zero together with the first derivative.

Definition 2.1. Let J be an signature operator on a Hilbert space $\mathfrak{H}$, $J = J^* = J^{-1}$. An operator T in $\mathfrak{H}$ is called J-selfadjoint if $JT = (JT)^*$.

It is clear that A is a J-selfadjoint operator. We will investigate the operator A in the framework of extension theory of symmetric operators. For this purpose we recall the following

Definition 2.2 ([1]). Let S be a closed symmetric operator with equal finite deficiency indices (n,n), $n < \infty$. A closed operator $\widetilde{S}$ is called a quasi-selfadjoint extension of S if

$$S \subset \widetilde{S} \subset S^* \qquad \text{and} \qquad \dim\left(\operatorname{dom}(\tilde{S}) / \operatorname{dom}(S)\right) = n.$$

Let $A_{min} := A \cap A^*$, $A^{\pm}_{min} := \pm L^{\pm}_{min}$. Then

$$A_{min} = A^-_{min} \oplus A^+_{min}, \qquad \operatorname{dom}(A_{min}) := \{y \in \operatorname{dom}(L) : \ y(0) = y'(0) = 0\}. \tag{2.3}$$

It is clear that A_{min} is a simple symmetric operator with deficiency indices $(2,2)$ and A is its quasi-selfadjoint extension. Indeed,

$$\operatorname{dom}(A) := \left\{y \in \operatorname{dom}\left((A^+_{min})^*\right) \oplus \left((A^-_{min})^*\right) : \ y(+0) = y(-0), \ y'(+0) = y'(-0)\right\}, \tag{2.4}$$

and $\dim(\operatorname{dom}(A)/\operatorname{dom}(A_{min})) = 2$.

Note in conclusion that if q is bounded, then $\operatorname{dom}(A) := \operatorname{dom}(L) = W^2_2(\mathbb{R})$, the Sobolev space, and $\operatorname{dom}(A_{min}) = W^{2,0}_2(\mathbb{R}) := \{y \in W^2_2(\mathbb{R}) : \ y(0) = y'(0) = 0\}$.

2.2 Weyl functions

Recall definition of the Weyl functions of the Sturm-Liouville operator (2.1) assuming as before, the limit point cases at $\pm\infty$. Denote by $s(x,\lambda)$ and $c(x,\lambda)$ the solutions of

$$-y''(x) + q(x)y(x) = \lambda y(x)$$

obeying the following initial conditions

$$s(0,\lambda) = \frac{d}{dx}c(0,\lambda) = 0, \qquad \frac{d}{dx}s(0,\lambda) = c(0,\lambda) = 1.$$

According to Weyl theory (see [39]) there exists the function $m_\pm(\lambda)$ on $\mathbb{C}_+ \cup \mathbb{C}_-$ such that

$$s(\cdot,\lambda) \mp m_\pm(\lambda)c(\cdot,\lambda) \in L^2(\mathbb{R}_\pm). \tag{2.5}$$

The function $m_\pm$ is called *the Weyl function of* $L^\pm_{min}$ corresponding to the initial condition $y'(0) = 0$. The functions

$$M_\pm(\lambda) := \pm m_\pm(\pm\lambda) \tag{2.6}$$

is said to be *the Weyl function of* $A^\pm_{min}$ (corresponding to the initial condition $y'(0) = 0$).

Define

$$\psi_\pm(\cdot,\lambda) := \begin{cases} -\left(s_\pm(\cdot,\pm\lambda) - M_\pm(\lambda)c(\cdot,\pm\lambda)\right), & x \in \mathbb{R}_\pm, \\ 0 & x \in \mathbb{R}_\mp. \end{cases} \tag{2.7}$$

It is easily seen that $\psi_\pm(\cdot,\lambda) \in L^2(\mathbb{R}_\pm)$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $(A^\pm_{min})^*\psi_\pm(x,\lambda) = \lambda\psi_\pm(x,\lambda)$.

Recall that a function $m(\lambda)$ is called *an R-function (Herglotz or Nevanlinna function)* [1, 24] if it is holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$,

$$\operatorname{Im}\lambda \cdot \operatorname{Im} m(\lambda) > 0 \quad \text{for} \ \ \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \qquad \text{and} \qquad m(\bar\lambda) = \overline{m(\lambda)}.$$

The set of all *R-functions* is denoted by (R) (see [24]).

The functions $m_\pm$, as well as $M_\pm$ are R-functions (see [39]). Moreover, it follows from (2.6) and the known integral representation of $m_\pm(\lambda)$ (see [38, 47]) that $M_\pm(\lambda)$ admits the following integral representation

$$M_\pm(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma_\pm(t)}{t-\lambda} \qquad \text{and} \qquad \int_{\mathbb{R}} \frac{d\Sigma_\pm(t)}{1+|t|} < \infty. \tag{2.8}$$

with a (nonunique) nondecreasing scalar function $\Sigma_\pm(t)$. Note that $\Sigma_\pm(t)$ in (2.8) is uniquely determined by the following normalized conditions:

$$2\Sigma_\pm(t) = \Sigma_\pm(t+0) + \Sigma_\pm(t-0)\ , \quad \Sigma_\pm(0) = 0.$$

Note also that (2.8) gives a holomorphic continuation of $m_\pm(\lambda)$ to $\mathbb{C} \setminus \operatorname{supp} d\Sigma_\pm$.

Moreover, the known asymptotic relations for $m_\pm(\cdot)$ (see [38]) yield

$$M_\pm(\lambda) = \pm\frac{i}{\sqrt{\pm\lambda}} + O\left(\frac{1}{\lambda}\right), \quad (\lambda \to \infty,\ 0 < \delta < \arg\lambda < \pi - \delta) \tag{2.9}$$

$$\Sigma_\pm(t) = \pm\frac{2}{\pi}\sqrt{\pm t}\ \pm \Sigma_\pm(\pm\infty)\ + o(1), \quad t \to \pm\infty. \tag{2.10}$$

Here and below $\sqrt{z}$ is the branch of the multifunction on the complex plane $\mathbb{C}$ with the cut along $\mathbb{R}_+$, singled out by the condition $\sqrt{-1} = i$. We assume that $\sqrt{\lambda} \geq 0$ for $\lambda \in [0, +\infty)$.

Consider the operator

$$A_0^{\pm} := (A_{min}^{\pm})^* \upharpoonright \mathrm{dom}(A_0^{\pm}), \qquad \mathrm{dom}(A_0^{\pm}) = \{y \in \mathrm{dom}\left((A_{min}^{\pm})^*\right) : \ y'(\pm 0) = 0\}. \tag{2.11}$$

Clearly, $A_0^{\pm} = (A_0^{\pm})^*$. The function $\Sigma_{\pm}$ is *the spectral function* of $A_0^{\pm}$ [39, 47]. It means that the generalized Fourier transform $\mathcal{F}_{\pm}$, defined by

$$(\mathcal{F}_{\pm} f)(t) := \operatorname*{l.i.m.}_{x_1 \to \pm\infty} \pm \int_0^{x_1} f(x) c(x, \pm t) dx, \tag{2.12}$$

is an isometric operator from $L^2(\mathbb{R}_{\pm})$ onto $L^2(\mathbb{R}, d\Sigma_{\pm})$. Here l.i.m. denotes the strong limit in $L^2(\mathbb{R}, d\Sigma_{\pm})$.

The operator $\widehat{A}_0^{\pm} := \mathcal{F}_{\pm} A_0^{\pm} \mathcal{F}_{\pm}^{-1}$ is the operator of multiplication by t in $L^2(\mathbb{R}, d\Sigma_{\pm}(t))$, $\widehat{A}_0^{\pm} : \ g(t) \to \ tg(t)$ (see [39, 47]). Note that $\sigma(A_0^{\pm}) = \operatorname{supp} d\Sigma_{\pm}$.

Suppose $f \in L^2(\mathbb{R})$. Let $f_{\pm} := P_{\pm} f \in L^2(\mathbb{R}_{\pm})$ where $P_{\pm}$ is the orthoprojection in $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_{\pm})$. The following two representations of the resolvent $\mathcal{R}_{A_0^{\pm}}$ are known (see [39, 47]):

$$(\mathcal{R}_{A_0^{\pm}}(\lambda) f_{\pm})(x) = \int_{\mathbb{R}} \frac{c(x, \pm t)\, (\mathcal{F}_{\pm} f_{\pm})(t)\, d\Sigma_{\pm}(t)}{t - \lambda}, \tag{2.13}$$

$$(\mathcal{R}_{A_0^{\pm}}(\lambda) f_{\pm})(x) = \mp \psi_{\pm}(x, \lambda) \int_0^{\pm x} c(s, \pm\lambda) f(s) ds \mp c(x, \pm\lambda) \int_{\pm x}^{\pm\infty} \psi_{\pm}(s, \lambda) f(s) ds. \tag{2.14}$$

2.3 Definitizable operators

The spectral theory of linear operators in Kreĭn spaces can be found in [3], [37]. Here we give some basic definitions.

Consider a Hilbert space $\mathfrak{H}$ with a scalar product $(\cdot, \cdot)$. Let J be an operator in $\mathfrak{H}$ such that $J = J^{-1} = J^*$. By $[\cdot, \cdot]$ we define a Hermitian sesquilinear form (s.f.) $(J\cdot, \cdot)$. Then the pair $\mathcal{K} = (\mathfrak{H}, [\cdot, \cdot])$ is a Kreĭn space (see the literature cited above). If $J \neq I$, then the s.f. $[\cdot, \cdot]$ is indefinite.

Let T be a closely defined operator in $\mathfrak{H}$. Then J-adjoint operator $T^{[*]}$ is defined by

$$[Tf, g] = [f, T^{[*]} g], \qquad f \in D(T), \ g \in D(T^{[*]}).$$

Clearly, $T^{[*]} = JT^*J$, where T^* is the adjoint operator with respect to the scalar product $(\cdot, \cdot)$. An operator T is called *J-selfadjoint* if $T = T^{[*]}$. Evidently, this definition is equivalent to Definition 2.1 and

$$T = T^{[*]} \quad \Longleftrightarrow T = JT^*J \ .$$

Definition 2.3 ([37]). A J-selfadjoint operator T is called definitizable if $\rho(T) \neq \emptyset$ and there exist a real polynomial p such that

$$[p(T) f, f] \geq 0 \quad \text{for} \quad f \in \mathrm{dom}(p(T)) \ .$$

Definitizable operators have spectral functions with critical points. Thus theirs spectral properties are close to spectral properties of selfadjoint operators in some sense (see [37]).

Operators of the form (2.2) are J-selfadjoint. In this case, $\mathfrak{H} = L^2(\mathbb{R})$ and J is a multiplication operator by $\operatorname{sgn} x$. Such operators can be nondefinitizable. The following theorem gives a criterion of definitizability.

Theorem 2.1 ([31, 32]). *Let $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ be an operator of the form (2.2). Then A is definitizable if and only if the sets $\operatorname{supp} d\Sigma_+$ and $\operatorname{supp} d\Sigma_-$ (see Subsection 2.2 for definitions) are separated by a finite number of points, i.e., there exists a finite ordered set*

$$\{\alpha_j\}_{j=1}^{2n-1}, \quad -\infty = \alpha_0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{2n-1} < \alpha_{2n} = +\infty,$$

such that

$$\operatorname{supp} d\Sigma_- \subset \bigcup_{k=0}^{n-1} [\alpha_{2k}, \alpha_{2k+1}], \qquad \operatorname{supp} d\Sigma_+ \subset \bigcup_{k=0}^{n-1} [\alpha_{2k+1}, \alpha_{2k+2}].$$

Several conditions of definitizability in abstract terms was given in [22] and [23].

Spectral properties of some classes of differential definitizable operators was studied in [7, 16, 10]; see also references in [10].

Definition 2.4. An operator T is called J-nonnegative if

$$[Tf, f] \ge 0 \quad \text{for} \quad f \in \operatorname{dom}(T)\ .$$

Denote the *root subspace* (the algebraic eigensubspace) of T for λ by $\mathfrak{L}_\lambda(T)$, that is

$$\mathfrak{L}_\lambda(T) := \operatorname{span}\{\ker(T-\lambda)^k :\ k \in \mathbb{Z}_+\}.$$

Proposition 2.2 ([51], see also [3]). *Let T be a J-nonnegative operator. Then*

(i) $\sigma_p(A) \cap (\mathbb{C}_+ \cup \mathbb{C}_-) = \emptyset$.

(ii) *If $\lambda \in \sigma_p(T)$ and $\lambda \ne 0$, then the eigenvalue λ is semisimple, i.e., $\mathfrak{L}_\lambda = \ker(T-\lambda)$.*

(iii) *If $0 \in \sigma_p(T)$, then $\mathfrak{L}_0 = \ker T^2$ (generally, $\mathfrak{L}_0 \ne \ker T$).*

2.4 Finite-zone potentials

Following [38] we recall a definition of Sturm-Liouville operator with a finite-zone potential. Let $N \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Consider sets of real numbers $\{\overset{l}{\mu}_j\}_{j=0}^{N+1}$, $\{\overset{r}{\mu}_j\}_0^N$, $\{\xi_j\}_1^N$ such that

$$-\infty = \overset{l}{\mu}_0 < \overset{r}{\mu}_0 < \overset{l}{\mu}_1 < \overset{r}{\mu}_1 < \cdots < \overset{l}{\mu}_N < \overset{r}{\mu}_N < \overset{l}{\mu}_{N+1} = +\infty,$$

$\xi_j \in [\overset{l}{\mu}_j, \overset{r}{\mu}_j]$, $j = 1, \ldots, N$. Define polynomials $R(\lambda)$, $P(\lambda)$ by

$$P(\lambda) = \prod_{j=1}^{N} (\lambda - \xi_j), \qquad R(\lambda) = (\lambda - \overset{r}{\mu}_0) \prod_{j=1}^{N} (\lambda - \overset{l}{\mu}_j)(\lambda - \overset{r}{\mu}_j).$$

Then there exist (see [38]) real polynomials $S(\lambda)$ and $Q(\lambda)$ of degrees $\deg S = N+1$ and $\deg Q = N-1$ respectively and such that

$$S(\lambda) = \prod_{j=0}^{N} (\lambda - \tau_j), \quad \tau_0 \in (-\infty, \overset{r}{\mu}_0], \quad \tau_j \in [\overset{l}{\mu}_j, \overset{r}{\mu}_j], \quad j \in \{1, \ldots, N\},$$

and such that the following identity holds

$$P(\lambda)S(\lambda) - Q^2(\lambda) = R(\lambda). \tag{2.15}$$

The function

$$m_\pm(\lambda) := \pm \frac{P(\lambda)}{Q(\lambda) \mp i\sqrt{R(\lambda)}} \tag{2.16}$$

is the Weyl function corresponding to the Neumann boundary value problem on $\mathbb{R}_\pm$ for some Sturm-Liouville operator $L = -d^2/dx^2 + q(x)$ with a bounded quasi-periodic potential $q = \bar{q}$. Here the branch of the multifunction is chosen in such a way that $m_\pm(\cdot)$ is R-function.

Definition 2.5. A (quasi-periodic) potential $q = \bar{q}$ is called a finite-zone potential if the Weyl functions $m_\pm$ of $L_\pm$ defined by (2.5) admit representations (2.16).

Assume q to be a finite-zone potential. Then q is an analytic function, and the nth derivative $\frac{d^n}{dx^n}q$ is bounded on $\mathbb{R}$ for any $n \in \mathbb{N}$. Moreover, the spectrum of $L = -d^2/dx^2 + q(x)$ is absolutely continuous, and

$$\sigma(L) = \sigma_{ac}(L) = [\overset{r}{\mu}_0, \overset{l}{\mu}_1] \cup [\overset{r}{\mu}_1, \overset{l}{\mu}_2] \cup \cdots \cup [\overset{r}{\mu}_N, +\infty).$$

Combining (2.16) with (2.6), we get

$$M_\pm(\lambda) = \frac{P(\pm\lambda)}{Q(\pm\lambda) \mp i\sqrt{R(\pm\lambda)}}. \tag{2.17}$$

Using (2.15), we rewrite (2.17) as

$$M_\pm(\lambda) = \frac{Q(\pm\lambda) \pm i\sqrt{R(\pm\lambda)}}{S(\pm\lambda)}. \tag{2.18}$$

2.5 Boundary triplets and abstract Weyl functions

2.5.1 Weyl functions and spectra of proper extensions.

Let $\mathfrak{H}$ and $\mathcal{H}$ be separable Hilbert spaces.

Definition 2.6. A closed linear relation Θ in $\mathcal{H}$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$.

Example 2.1. For any closed operator B in $\mathcal{H}$ its graph $G(B)$ is a closed relation in $\mathcal{H}$.

Let S be a closed densely defined symmetric operator in $\mathfrak{H}$ with equal deficiency indices $n_+(S) = n_-(S)$, where $n_\pm(S) := \dim \mathfrak{N}_{\pm i}$ and $\mathfrak{N}_\lambda := \ker(S^* - \lambda)$.

Definition 2.7 ([1]). A closed extension $\tilde{S}$ of S is called a proper extension if $S \subset \tilde{S} \subset S^*$. The set of all proper extensions is denoted by Ext_S.

Recall the definition of a boundary triplet.

Definition 2.8 ([19]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_j : \ \mathrm{dom}(S^*) \longrightarrow \mathcal{H}, \quad j \in \{0, 1\}$, is called a boundary triplet for the operator S^* if the following conditions are satisfied:

(i) The second Green formula

$$(S^*f, g) - (f, S^*g) = (\Gamma_1 f, \Gamma_0 g)_\mathcal{H} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{H}, \qquad f, g \in \mathrm{dom}(S^*), \tag{2.19}$$

holds;

(ii) The mapping $\Gamma : \ \mathrm{dom}(S^*) \longrightarrow \mathcal{H} \oplus \mathcal{H}, \ \Gamma f := \{\Gamma_0 f, \Gamma_1 f\}$ is surjective.

Definition 2.8 allows one to describe the set Ext_S in the following way (see [11, 12]).

Proposition 2.3 ([11, 12]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. Then the mapping Γ establishes a bijective correspondence $\tilde{S} \to \Theta := \Gamma(\mathrm{dom}(\tilde{S}))$ between the set Ext_S and the set of closed linear relations in $\mathcal{H}$.*

By Proposition 2.3 the following definition is natural.

Definition 2.9. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator S^*.

(*i*) Denote $S_\Theta = \tilde{S}$, if $\Theta = \Gamma(\mathrm{dom}(\tilde{S}))$ that is

$$S_\Theta := S^*|D_\Theta, \quad \text{where} \quad \mathrm{dom}(S_\Theta) = D_\Theta := \{f \in \mathrm{dom}(S^*) : \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}. \tag{2.20}$$

(*ii*) If $\Theta = G(B)$ is the graph of $B \in \mathcal{C}(\mathcal{H})$ then $\mathrm{dom}(S_\Theta)$ determined by the equation $\mathrm{dom}(S_B) = D_B := D_\Theta = \ker(\Gamma_1 - B\Gamma_0)$. We set $S_B := S_\Theta$.

Let us make the following remarks.

Remark 2.1. 1) The deficiency indices $n_\pm(S)$ are equal to the dimension of $\mathcal{H}$, i.e., $\dim(\mathcal{H}) = n_\pm(S)$.

2) There exist two self-adjoint extensions $S_j := S^*|\ker(\Gamma_j)$ which are naturally associated to a boundary triplet. According to Definition 2.9 $S_j = S_{\Theta_j}, j \in \{0, 1\}$, where $\Theta_0 = \{0\} \times \mathcal{H}$, $\Theta_1 = \mathcal{H} \times \{0\}$. Conversely, if S_0 is a self-adjoint extension of A, then there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $S_0 = S^*|\ker(\Gamma_0)$.

3) Θ is the graph of a closed operator B iff $\tilde{S}$ and S_0 are disjoint, i.e., $\mathrm{dom}(\tilde{S}) \cap \mathrm{dom}(S_0) = \mathrm{dom}(S)$.

4) $\Theta = G(B)$ with $B \in [\mathcal{H}]$ iff $\tilde{S}$ and S_0 are transversal, i.e., $\tilde{S}$ and S_0 are disjoint and $\mathrm{dom}(\tilde{S}) + \mathrm{dom}(S_0) = \mathrm{dom}(S^*)$.

Definition 2.10 ([13]). A proper extension $\tilde{S} \in \mathrm{Ext}_S$ is called an almost solvable if there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and an operator $B \in [\mathcal{H}]$ such that

$$\mathrm{dom}(\tilde{S}) = \mathrm{dom}(S_B) := \ker(\Gamma_1 - B\Gamma_0). \tag{2.21}$$

The set of almost solvable extensions is denoted by $\mathcal{A}s_S$. Note that the class $\mathcal{A}s_S$ is sufficiently wide. Proper extensions having two regular points $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\mathrm{Im}\,\lambda_1 \cdot \mathrm{Im}\,\lambda_2 < 0$ belong to $\mathcal{A}s_S$. All quasiselfadjoint extensions are in $\mathcal{A}s_S$.

In [11, 12] the concept of Weyl function was generalized to an arbitrary symmetric operator T with infinite deficiency indices $n_+(A) = n_-(A)$. Recall some basic facts about Weyl functions.

Definition 2.11 ([11, 12]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. The Weyl function of T corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unique mapping

$$M(\cdot): \ \rho(T_0) \longrightarrow [\mathcal{H}] \tag{2.22}$$

satisfying

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda = \ker(S^* - \lambda I), \quad \lambda \in \rho(S_0). \tag{2.23}$$

It is well known (see [11, 12]) that the above implicit definition of the Weyl function is correct and $M(\cdot)$ is an operator-valued R-function obeying $0 \in \rho(\mathrm{Im}(M(i)))$(see [14]). The Weyl function immediately provides some information about the "spectral properties" of proper extensions. We confine ourselves to the case of almost solvable extensions of the symmetric operator S.

Proposition 2.4 ([12, 13]). *Suppose that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^*, $M(\cdot)$ is the corresponding Weyl function, $\lambda \in \rho(S_0)$ and $B \in [\mathcal{H}]$. Then:*

1) $\lambda \in \rho(S_B)$ *if and only if* $0 \in \rho(B - M(\lambda))$;

2) $\lambda \in \sigma_i(S_B)$ *if and only if* $0 \in \sigma_i(B - M(\lambda))$, $i \in \{p, r, c\}$.

We demonstrate applicability of Proposition 2.4 by describing a discrete spectrum of the operator A.

Proposition 2.5. *Let $S := A_{\min}$ be a (minimal) symmetric operator defined by (2.3) and let $M_\pm(\cdot)$ be defined by (2.6). Then*

(i) $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ *defined by*

$$\Gamma_0,\ \Gamma_1 :\ \operatorname{dom}(A^*_{\min}) \to \mathcal{H} = \mathbb{C}^2, \qquad \Gamma_0 f = \begin{pmatrix} f(+0), \\ f'(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f'(+0) \\ -f(-0) \end{pmatrix}, \tag{2.24}$$

forms a boundary triplet for the operator $S^* = A^*_{\min}$;

(ii) *The corresponding Weyl function is*

$$M(\lambda) := M_\Pi(\lambda) = \operatorname{diag}\bigl(-M_+^{-1}(\lambda), M_-(\lambda)\bigr); \tag{2.25}$$

(iii) *The operator $A = JL$ defined by (2.2) is a quasi-selfadjoint extension of S and it is determined by*

$$A = S^*|\operatorname{dom} A, \qquad \operatorname{dom} A = \ker(\Gamma_1 - B\Gamma_0), \qquad \text{where} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.26}$$

that is $A = S_B$;

(iv) *$\rho(A) \neq \emptyset$ and $\lambda_0 \in \rho(A) \cap \mathbb{C}_\pm$ if and only if $M_+(\lambda_0) \neq M_-(\lambda_0)$. Moreover, $\rho(A) \cap \mathbb{R} = \cup_j(\alpha_j, \beta_j)$ where (α_j, β_j) is such an interval that both M_+ and M_- admit holomorphic continuation trough (α_j, β_j) and $M_+(x + i0) \neq M_-(x + i0)$, $x \in (\alpha_j, \beta_j)$.*

(v) *The sets $\sigma_p(A) \cap \mathbb{C}_\pm$ are at most countable with possible limit points belonging to $\mathbb{R} \cup \{\infty\}$. Moreover, $\lambda_0 \in \sigma_p(A) \cap \mathbb{C}_\pm$ if and only if $M_+(\lambda_0) = M_-(\lambda_0)$. In the latter case $\dim \mathfrak{L}_{\lambda_0}(A) = \mathfrak{m}(\lambda_0)$, where $\mathfrak{m}(\lambda_0)$ is the multiplicity of λ_0 as a zero of the analytic function $M_+(\lambda) - M_-(\lambda)$;*

(vi) *The spectrum $\sigma(A)$ is symmetric with respect to the real line, that is $\lambda_0 \in \sigma_p(A) \Longleftrightarrow \overline{\lambda}_0 \in \sigma_p(A)$ and $\dim \mathfrak{L}_{\lambda_0}(A) = \dim \mathfrak{L}_{\overline{\lambda}_0}(A)$ (equivalently $\lambda_0 \in \sigma(A) \Longleftrightarrow \lambda_0 \in \sigma(A^*)$ and $\dim \mathfrak{L}_{\lambda_0}(A) = \dim \mathfrak{L}_{\lambda_0}(A^*)$.*

Proof. (i)-(iii) These statements are obvious.

(iv) By Proposition 2.4 $\lambda_0 \in \rho(A)$ if and only if $0 \in \rho(B - M(\lambda_0)$, that is

$$\det\bigl(B - M(\lambda)\bigr) = \det \begin{pmatrix} M_+^{-1}(\lambda) & 1 \\ -1 & -M_-(\lambda) \end{pmatrix} = M_+^{-1}(\lambda) \cdot [M_+(\lambda) - M_-(\lambda)] \neq 0. \tag{2.27}$$

Note that due to (2.9) $M_+(\cdot)$ and $M_-(\cdot)$ have different asymptotic behavior along any semi-axes $t \cdot e^{i\varphi}$, $t > 0$ with $\varphi \in (0, \pi/2)$. Hence $M_+ - M_- \not\equiv 0$, that is the determinant $\det\bigl(B - M(\lambda)\bigr)$ does not vanish identically and $\rho(A) \neq \emptyset$.

The last statement follows from Proposition 2.4 and the identity

$$(B - M(\lambda))^{-1} = \frac{1}{M_+(\lambda) - M_-(\lambda)} \begin{pmatrix} 1 & M_+(\lambda) \\ -M_+(\lambda) & -M_+(\lambda)M_-(\lambda) \end{pmatrix}$$

(v) By Proposition 2.4 $\sigma(S_B) \cap \mathbb{C}_\pm$ coincides with the set of zeros of the determinant $\det\big(B - M(\lambda)\big)$ in $\mathbb{C}_\pm$. Due to (2.27) $\sigma(S_B) \cap \mathbb{C}_\pm$ coincides with the set of zeros of $M_+(\lambda) - M_-(\lambda)$ in $\mathbb{C}_\pm$ since $M_+(\lambda)$ has no zeros in $\mathbb{C}_\pm$. The analytic function $M_+(\lambda) - M_-(\lambda)$ does not vanish identically, hence it has at most countable set of zeros in both $\mathbb{C}_+$ and $\mathbb{C}_-$. The rest statements follow from analyticity of $M_+ - M_-$ and Proposition 2.4.

(vi) Note that $M_+(\lambda_0) - M_-(\lambda_0) = 0$ yields $M_+(\overline{\lambda}_0) - M_-(\overline{\lambda}_0) = \overline{M_+(\lambda_0) - M_-(\lambda_0)} = 0$. Similar implication is valid for jth derivative. This completes the proof. $\square$

2.5.2 A functional model of a symmetric operator.

Next we recall construction of a functional model of a symmetric operator following [14], [42]. We need only the case of the deficiency indices $(1,1)$.

Let $\Sigma(t)$ be a nondecreasing scalar function obeying the conditions

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma(t) < \infty, \quad \int_{\mathbb{R}} d\Sigma(t) = \infty\ , \quad \Sigma(t) = \frac{1}{2}(\Sigma(t-0) + \Sigma(t+0)), \quad \Sigma(0) = 0. \tag{2.28}$$

The operator of multiplication $Q_\Sigma : f(t) \to tf(t)$ is selfadjoint in $L^2(\mathbb{R}, d\Sigma)$. Consider its restriction

$$\widehat{T}_\Sigma = Q_\Sigma \upharpoonright \operatorname{dom}(\widehat{T}_\Sigma), \qquad \operatorname{dom}(\widehat{T}_\Sigma) = \{f \in \operatorname{dom} Q_\Sigma : \int_{\mathbb{R}} f(t) d\Sigma(t) = 0\}.$$

Then $\widehat{T}_\Sigma$ is a simple densely defined symmetric operator in $L^2(\mathbb{R}, d\Sigma)$ with deficiency indices (1,1). The adjoint operator $\widehat{T}_\Sigma^*$ has the form

$$\operatorname{dom}(\widehat{T}_\Sigma^*) = \{f = f_Q + t(t^2+1)^{-1}h \ : \ f_Q \in \operatorname{dom}(Q_\Sigma),\ h \in \mathbb{C}\}, \quad \widehat{T}_\Sigma^* f = tf_Q - (t^2+1)^{-1}h.$$

Let $C \in \mathbb{R}$. Define linear mappings $\Gamma_0^\Sigma, \Gamma_1^{\Sigma,C} :\ \operatorname{dom}(\widehat{T}_\Sigma^*) \to \mathbb{C}$ by

$$\Gamma_0^\Sigma f = h, \qquad \Gamma_1^{\Sigma,C} f = Ch + \int_{\mathbb{R}} f_Q(t) d\Sigma(t), \tag{2.29}$$
$$\text{where} \qquad f = f_Q + t(t^2+1)^{-1}h \in \operatorname{dom}(\widehat{T}_\Sigma^*), \qquad f_Q \in \operatorname{dom}(Q_\Sigma), \qquad h \in \mathbb{C}.$$

Then $\{\mathbb{C}, \Gamma_0^\Sigma, \Gamma_1^{\Sigma,C}\}$ is a boundary triple for $\widehat{T}_\Sigma^*$. The function

$$M_{\Sigma,C}(\lambda) := C + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \operatorname{supp} d\Sigma, \tag{2.30}$$

is the corresponding Weyl function of $\widehat{T}_\Sigma$.

2.6 Some facts of Hardy spaces theory

2.6.1 The Hilbert transform in weighted spaces

Let us recall some facts of Hardy spaces theory following [18] and [36].

Let μ be a Borel measure on $\mathbb{R}$ obeying $\int_{\mathbb{R}}(1+t^2)^{-1}d\mu(t) < \infty$. As usual we denote by $u(\lambda) = \mathcal{P}_\lambda(\mu)$ its harmonic extension (the Poisson integral) at the point $\lambda = x+iy \in \mathbb{C}_+$,

$$u(x+iy) := \mathcal{P}_\lambda(\mu) := (P_y * \mu)(x) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} d\mu(t). \tag{2.31}$$

For any function $\varphi \in L^1(dt/1+t^2)$ we put $\mathcal{P}_\lambda(\varphi) := \mathcal{P}_\lambda(\mu)$ where $\mu = \varphi dx$.

Moreover, assuming that $\int_{\mathbb{R}}(1+|t|)^{-1}d\mu(t) < \infty$ one introduces the harmonic conjugate $\widetilde{u}(\cdot)$ of $u(\cdot)$ by setting

$$\widetilde{u}(x+iy) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-t}{(x-t)^2+y^2} d\mu(t). \tag{2.32}$$

Here we require the normalization $\lim_{y\to+\infty}\widetilde{u}(x+iy) = 0$. By Fatou theorem for a.e. $x \in \mathbb{R}$ the limit $\lim_{y\to 0} u(x,y) =: u(x+i0)$ exists and $u(x+i0) = \mu'(x)$. Moreover, the limit $\lim_{y\to 0}\widetilde{u}(x+iy) =: \widetilde{u}(x+i0)$ exists a.e. and coincides with the Hilbert transform of μ, that is

$$\widetilde{u}(x+i0) = (H\mu)(x) := \frac{1}{\pi}\lim_{\delta\to 0}\int\limits_{|x-t|>\delta} \frac{1}{x-t} d\mu(t). \tag{2.33}$$

If $f \in L^p(\mathbb{R})$ with $p \in [1,\infty)$, then by definition $(Hf)(x) := (H\mu)(x)$ with $\mu = fdx$. The operator H is a unitary operator on $L^2(\mathbb{R})$.

Recall the Helson - Szegö theorem [21] (see also [18]).

Theorem 2.6 (Helson, Szegö). *Let $d\mu$ be a positive Borel measure on $\mathbb{R}$, finite on compact sets. There is a constant K such that*

$$\int_{\mathbb{R}} |Hf(x)|^2 d\mu(x) \le K \int_{\mathbb{R}} |f(x)|^2 d\mu(x)$$

for all $f \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}, d\mu)$ if and only if μ is absolutely continuous, $d\mu(x) = w(x)dx$, and

$$\log w(x) = u + Hv, \quad u \in L^\infty(\mathbb{R}), \quad \|v\|_{L^\infty(\mathbb{R})} < \pi/2. \tag{2.34}$$

Theorem 2.6, the Helson-Szegö theorem, provides a necessary and sufficient condition for the Hilbert transform to be bounded on $L^2(d\mu)$.

Another solution to this problem has been obtained by Muckenhoupt [43] and Hunt, Muckenhoupt and Wheeden [44].

Theorem 2.7 (Hunt, Muckenhoupt, Wheeden). *Let $d\mu$ be a positive Borel measure on $\mathbb{R}$, finite on compact sets. Then the inequality*

$$\int_{\mathbb{R}} |Hf(x)|^2 d\mu(x) \le K_2 \int_{\mathbb{R}} |f(x)|^2 d\mu(x)$$

with K_2 independent of $f \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}, d\mu)$ holds if and only if $d\mu(x) = w(x)dx$ and the density $w(x)$ satisfies the following condition

$$\sup_{\mathcal{I}} \left(\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} w(t)dt\right)\left(\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}}\left(\frac{1}{w(t)}\right)dt\right) < \infty. \tag{2.35}$$

Here, in (2.35) sup is taken over the set of all (closed) intervals $\mathcal{I} \subset \mathbb{R}$.

Condition (2.35) is called the (A_2)-condition; we will write $w \in (A_2)$ if (2.35) is satisfied.

It is well known that the necessary part of the condition (2.35) remains valid (with the same proof) for two-weight estimates of Hilbert transform.

More precisely, suppose that $w_1(\cdot)$ and $w_2(\cdot)$ are two nonnegative functions (weights) and $E = \operatorname{supp} w_2 = \overline{E}$ is a topological support of w_2. Then the two-weight inequality

$$\int_{\mathbb{R}} |Hf(x)|^2 \cdot w_1(x)dx \le K_2 \int_{\mathbb{R}} |f(x)|^2 \cdot w_2(x)dx \tag{2.36}$$

implies the estimate

$$\sup_{\mathcal{I}} \left(\frac{1}{|\mathcal{I} \cap E|} \int_{\mathcal{I}} w_1(t)dt \right) \left(\frac{1}{|\mathcal{I} \cap E|} \int_{\mathcal{I}} \left(\frac{1}{w_2(t)} \right) dt \right) < \infty. \tag{2.37}$$

In turn, inequality (2.37) yields

$$\operatorname{vrai\,sup}_{t \in E} [w_1(x) \cdot w_2(x)^{-1}] = C < \infty. \tag{2.38}$$

In fact, inequalities (2.37) and (2.38) are not equivalent, that is (2.37) is stronger than (2.38).

Following [49] we mention one more consequence of two-weight estimate (2.36).

Proposition 2.8. *Let $w_1, w_2 \ge 0$ be two nonnegative measurable functions on $\mathbb{R}$ and $w_2^{-1}(\cdot)$ is finite a.e. on $\mathbb{R}$. Then for the two-weight estimate* (2.36) *to be valid it is necessary that*

$$\sup_{\lambda \in \mathbb{C}_+} \mathcal{P}_\lambda(w_1) \cdot \mathcal{P}_\lambda(w_2^{-1}) = C < \infty. \tag{2.39}$$

D. Sarason has conjectured that the converse is also true, that is condition (2.39) is also sufficient for the two-weight estimate to be hold. Later on F. Nazarov (see [49]) shown that it is false.

It is easily seen (and well known) that condition (2.39) is stronger than (2.37). Indeed, if x is a middle of $\mathcal{I}$, $y = |\mathcal{I}|/2$ and $\lambda = x + iy$, then $|\mathcal{I}|^{-1}\chi_{\mathcal{I}}(t) \le \pi P_y(x-t)$ (cf. [18, Theorem VI.1.2]). Hence for any nonnegative $\varphi \in L^1_{loc}(\mathbb{R})$

$$\frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} \varphi(t)dt \le \int_{\mathcal{I}} P_y(x-t)\varphi(t)dt = \mathcal{P}_\lambda(\varphi). \tag{2.40}$$

Also we will use the following result.

Proposition 2.9 (cf. Theorem 4 in [21]). *Let $\{t_j\}_{j=1}^N$ be a finite set of real numbers. Assume that a (positive) weight function $w(t)$, $t \in \mathbb{R}$, has the following properties:*

$$w(t) \asymp t^{\alpha_\infty} \qquad (|t| \to \infty), \qquad \text{where} \quad -1 < \alpha_\infty < 1, \tag{2.41}$$

$$w(t) \asymp |t - t_j|^{\alpha_j} \qquad (t \to t_j), \qquad \text{where} \quad -1 < \alpha_j < 1, \quad j = 1, \dots, N, \tag{2.42}$$

$$w(t) \asymp 1 \qquad (t \to t_0) \qquad \forall t_0 \in \mathbb{R} \setminus \{t_j\}_{j=1}^N. \tag{2.43}$$

Then $w \in (A_2)$, i.e., the weight function w satisfies (2.35) *with $p = 2$.*

Proof. In this proof the letter C will be used to denote a positive constant not necessarily the same at each occurrence.

If $w \notin (A_2)$, then there exists a sequence of intervals $\mathcal{I}_n = [a_n, b_n]$, $n \in \mathbb{N}$, with the following properties:

(S1) $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are monotone;

(S2) there exist limits $a = \lim a_n, \quad b = \lim b_n, \quad -\infty \le a \le b \le +\infty$;

(S3) $\lim\limits_{n\to\infty} \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} w(t)dt \right) \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} \frac{1}{w(t)} dt \right) = \infty.$

Let us suppose now that assumptions (2.41)-(2.43) hold true and let the sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ have properties (S1), (S2). We will prove that property (S3) does not hold in this case, i.e.,

$$\mathfrak{P}_n := \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} w(t)dt \right) \left(\frac{1}{|\mathcal{I}_n|} \int_{\mathcal{I}_n} \frac{1}{w(t)} dt \right) < C \qquad \text{for all} \quad n \in N. \tag{2.44}$$

First note that assumptions (2.41)- (2.43) yields that $w(\cdot) \in L^1_{loc}(\mathbb{R})$ and $\frac{1}{w(\cdot)} \in L^1_{loc}(\mathbb{R})$. Hence it suffices to show (2.44) for sufficiently large n.

We should consider 7 cases.

Case 1. Let $a = b = +\infty$ (the case $a = b = -\infty$ is similar).

By (2.41), $w(t) < C|t|^{\alpha_\infty}$ and $\frac{1}{w(t)} < C|t|^{-\alpha_\infty}$ for sufficiently large $t > 0$. Hence, for n large enough, we have

$$\mathfrak{P}_n = \frac{1}{(b_n - a_n)^2} \int_{a_n}^{b_n} w(t)dt \int_{a_n}^{b_n} \frac{1}{w(t)} dt < C \frac{1}{(b_n - a_n)^2} \int_{a_n}^{b_n} t^{\alpha_\infty} dt \int_{a_n}^{b_n} t^{-\alpha_\infty} dt.$$

Since $\alpha_\infty \in (-1, 1)$, we have

$$\mathfrak{P}_n < C \frac{(b_n^{1+\alpha_\infty} - a_n^{1+\alpha_\infty})(b_n^{1-\alpha_\infty} - a_n^{1-\alpha_\infty})}{(b_n - a_n)^2 (1 + \alpha_\infty)(1 - \alpha_\infty)} < C \frac{b_n^2 + a_n^2 - b_n^{1-\alpha_\infty} a_n^{1+\alpha_\infty} - b_n^{1+\alpha_\infty} a_n^{1-\alpha_\infty}}{b_n^2 + a_n^2 - 2b_n a_n}$$

(it is assumed that $a_n, b_n > 0$). By the Cauchy inequality,

$$b_n^{1-\alpha_\infty} a_n^{1+\alpha_\infty} + b_n^{1+\alpha_\infty} a_n^{1-\alpha_\infty} > 2b_n a_n.$$

Thus $\mathfrak{P}_n < C$ for n large enough.

Case 2. Let $a = -\infty$, $b = +\infty$.

By (2.41), there exist a constants $a_0 < 0$ and $b_0 > 0$ such that

$$w(t) < C|t|^{\alpha_\infty} \quad \text{and} \quad \frac{1}{w(t)} < C|t|^{-\alpha_\infty} \qquad \text{for} \quad t \in (-\infty, a_0) \cup (b_0, +\infty).$$

Therefore,

$$\mathfrak{P}_n < C \frac{1}{(b_n - a_n)^2} \left(\int_{a_n}^{a_0} |t|^{\alpha_\infty} dt + \int_{a_0}^{b_0} w(t)dt + \int_{b_0}^{b_n} t^{\alpha_\infty} dt \right) \times$$
$$\times \left(\int_{a_n}^{a_0} |t|^{-\alpha_\infty} dt + \int_{a_0}^{b_0} \frac{1}{w(t)} dt + \int_{b_0}^{b_n} t^{-\alpha_\infty} dt \right)$$

for n large enough. Taking into account the fact that $\int_{a_0}^{b_0} w(t)dt < \infty$ and $\int_{a_0}^{b_0} \frac{1}{w(t)} dt < \infty$, we get

$$\mathfrak{P}_n < C \frac{\left(|a_n|^{1+\alpha_\infty} - |a_0|^{1+\alpha_\infty} + C + b_n^{1+\alpha_\infty} - b_0^{1+\alpha_\infty}\right)\left(|a_n|^{1-\alpha_\infty} - |a_0|^{1-\alpha_\infty} + C + b_n^{1-\alpha_\infty} - b_0^{1-\alpha_\infty}\right)}{(b_n - a_n)^2}$$
$$< C \frac{(|a_n|^{1+\alpha_\infty} + b_n^{1+\alpha_\infty})(|a_n|^{1-\alpha_\infty} + b_n^{1-\alpha_\infty})}{(b_n - a_n)^2} < C.$$

Case 3. Let $-\infty < a = b < +\infty$, $a_n \uparrow a$, and $b_n \downarrow a(= b)$.
By (2.42)-(2.43), there exist $\alpha \in (-1, 1)$ such that

$$w(t) \asymp |t-a|^{\alpha}, \quad \frac{1}{w(t)} \asymp |t-a|^{-\alpha}, \qquad (t \to a).$$

So, for n large enough,

$$\begin{aligned}
\mathfrak{P}_n &< C\frac{1}{(b_n-a_n)^2}\left(\int_{a_n}^{a}|t-a|^{\alpha}dt + \int_{a}^{b_n}(t-a)^{\alpha}dt\right)\left(\int_{a_n}^{a}|t-a|^{-\alpha}dt + \int_{a}^{b_n}(t-a)^{-\alpha}dt\right)\\
&< C\,\frac{\left((a-a_n)^{1+\alpha}+(b_n-a)^{1+\alpha}\right)\left((a-a_n)^{1-\alpha}+(b_n-a)^{1-\alpha}\right)}{\left((b_n-a)+(a-a_n)\right)^2}\\
&= C\,\frac{(a-a_n)^2+(b_n-a)^2+(a-a_n)^{1-\alpha}(b_n-a)^{1+\alpha}+(a-a_n)^{1+\alpha}(b_n-a)^{1-\alpha}}{(a-a_n)^2+(b_n-a)^2+2(a-a_n)(b_n-a)}\\
&< C + C\,\frac{(a-a_n)^{1-\alpha}(b_n-a)^{1+\alpha}+(a-a_n)^{1+\alpha}(b_n-a)^{1-\alpha}}{\max\{(a-a_n)^2,(b_n-a)^2\}} < C.
\end{aligned}$$

Case 4. Let $-\infty < a < b = +\infty$ and $a_n \downarrow a$ (the case $-\infty = a < b < +\infty$, $b_n \uparrow b$ is similar).
By (2.41)-(2.43),

$$w(t) < Ct^{\alpha_\infty}, \quad \frac{1}{w(t)} < Ct^{-\alpha_\infty} \qquad \text{for} \quad t \in (b_0, +\infty), \tag{2.45}$$

where b_0 is a certain positive constant. Since

$$\int_{a_n}^{b} w(t)dt \le \int_{a}^{b} w(t)dt < C \qquad \text{and} \qquad \int_{a_n}^{b}\frac{1}{w(t)}dt \le \int_{a}^{b}\frac{1}{w(t)}dt < C$$

for all $n \in N$, we clearly have

$$\begin{aligned}
\mathfrak{P}_n &< \frac{1}{(b_n-a_n)^2}\left(\int_{a_n}^{b_0} w(t)dt + \int_{b_0}^{b_n} w(t)dt\right)\left(\int_{a_n}^{b_0}\frac{1}{w(t)}dt + \int_{b_0}^{b_n}\frac{1}{w(t)}dt\right)\\
&< C\frac{1}{(b_n-a_n)^2}\left(C + \int_{b}^{b_n} t^{\alpha_\infty}dt\right)\left(C + \int_{b}^{b_n} t^{-\alpha_\infty}dt\right)\\
&< C\frac{(b_n^{1+\alpha_\infty}-b_0^{1+\alpha_\infty})(b_n^{1-\alpha_\infty}-b_0^{1+\alpha_\infty})}{b_n^2-2b_na_n+a_n^2}.
\end{aligned}$$

It follows from $\lim b_n = +\infty$ that $\mathfrak{P}_n < C$ for $n \in N$.
In the same way one can treat the following cases:

Case 5: $-\infty < a = b < +\infty$, $a_n \downarrow a$, and $b_n \downarrow a(= b)$ (the case $a_n \uparrow a$, $b_n \uparrow a$ is similar);

Case 6: $-\infty < a < b = +\infty$, $a_n \uparrow a$ (the case $-\infty = a < b < +\infty$, $b_n \downarrow b$ is analogous);

Case 7: $-\infty < a < b < +\infty$.

Thus property (S3) does not hold. This shows that $w \in (A_2)$.

□

2.6.2 The Smirnov class

We denote by $\mathcal{N}^+(\mathbb{C}_+)$ the Smirnov class on $\mathbb{C}_+$. Recall that $\mathcal{N}^+(\mathbb{C}_+)$ consists of holomorphic on $\mathbb{C}_+$ functions $U(z)$ such that $U(z)$ admits the factorization

$$U(z) = c\, B(z)\, F(z)\, S(z), \quad z \in \mathbb{C}_+,$$

where B is a Blaschke product, F is an outer function, S is a singular function, c is a constant, $|c| = 1$ (see [18, Corollary II.5.6 and Theorem II.5.5]).

The following lemmas are well known.

Lemma 2.10. *If $f, g \in \mathcal{N}^+(\mathbb{C}_+)$, then $f + g \in \mathcal{N}^+(\mathbb{C}_+)$.*

Lemma 2.11. *Let $\{t_j\}_{j=1}^N$ be a finite set of real numbers. Let $U(z)$ be a holomorphic function on $\mathbb{C}_+$ such that*

$$\begin{gathered} U(z) = O(z^{\alpha_\infty}) \quad (z \to \infty), \\ U(z - t_j) \asymp |z - t_j|^{\alpha_j} \quad (z \to t_j),\ j = 1, \ldots, N, \\ U(z - z_0) = O(1) \quad (z \to z_0) \quad \forall z_0 \in (\mathbb{C}_+ \cup \mathbb{R}) \setminus \{t_j\}_{j=1}^N, \end{gathered}$$

where $\alpha_\infty \in \mathbb{R}_+$, $\alpha_j \in \mathbb{R}_-$, $j = 1, \ldots, N$. Then $U(z) \in \mathcal{N}^+(\mathbb{C}_+)$.

The proofs of these lemmas are standard.

3 Similarity conditions

3.1 Characteristic functions and similarity

Let S be a symmetric operator in a Hilbert space $\mathfrak{H}$ with finite deficiency indices (n, n), $n \in \mathbb{N}$. Let T be a quasi-selfadjoint extension of S. Then (see Subsection 2.5 and [14]) there exists a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for T^*_{min} such that $\operatorname{dom} T = \ker(\Gamma_1 - B\Gamma_0)$ with some $B \in [\mathcal{H}]$, that is $T = S_B$. Let $M(\cdot)$ be the Weyl function associated with the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. The characteristic function $\theta_T(\cdot)$ of almost solvable extension $T(\in \operatorname{Ext}_S)$ is determined and investigated in [13], [14]. In the sequel we need the following formula for the characteristic function $\theta_T(\cdot)$ obtained in [13]. It express the $\theta_T(\cdot)$ by means of a boundary operator B and the corresponding Weyl function $M(\lambda)$.

Theorem 3.1 ([13]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^*, $M(\cdot)$ the corresponding Weyl function, $B \in [\mathcal{H}]$, and E an auxiliary Hilbert space. Then for any factorization $B_I := (B - B^*)/2i = K\mathcal{J}K^*$ of B_I with $K \in [E, \mathcal{H}]$ and $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1} \in [E]$, the characteristic function $\theta(\lambda) := \theta_{A_B}(\lambda)$ of the extension $A_B(\in \operatorname{Ext}_S)$, $\operatorname{dom} S_B = \ker(\Gamma_1 - B\Gamma_0)$, admits the following representation*

$$\theta_T(\lambda) = I + 2iK^*\big(B^* - M(\lambda)\big)^{-1} K\mathcal{J}. \tag{3.1}$$

It is shown in [13] that if $\ker(B - B^*) = \{0\}$, then

$$\theta_T(\lambda) = (B - M(\lambda))\,(B^* - M(\lambda))^{-1}.$$

It is well known that the characteristic function $\theta_T(\lambda)$ obeys the following properties ($\mathcal{J}$-properties):

$$\begin{cases} \omega_\theta(\lambda) := \mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) \ > \ 0, & \lambda \in \mathbb{C}_+, \\ \omega_\theta(\lambda) := \mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) \ < \ 0, & \lambda \in \mathbb{C}_-. \end{cases} \tag{3.2}$$

The second $\mathcal{J}$-form $\omega_{\theta^*}(\lambda) := \mathcal{J} - \theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)$ has the same properties.

Next we recall some (sufficient) conditions of similarity to a selfadjoint operator in terms of the characteristic function $\theta_T(\lambda)$ and the corresponding $\mathcal{J}$-forms $\omega_\theta(\cdot)$ and $\omega_{\theta^*}(\cdot)$.

Theorem 3.2 ([41]). *Let T be a solvable extension of S, that is $\operatorname{dom} T = \ker(\Gamma_1 - B\Gamma_0)$, with $B \in [\mathcal{H}]$, $B_I := (B - B^*)/2i = K\mathcal{J}K^*$ where $\mathcal{J} := \operatorname{sgn} B_I$ and $\pi_\pm := (I \pm \mathcal{J})/2$. Suppose that $\sigma(T) \subset \mathbb{R}$ and at least one of the following two conditions is satisfied*

$$(i) \qquad \max\left\{\sup_{\lambda\in\mathbb{C}_-} \|\pi_+\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\pi_+\|, \qquad \sup_{\lambda\in\mathbb{C}_+} \|\pi_-\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\pi_-\|\right\} < \ \infty. \tag{3.3}$$

$$(ii) \qquad \max\left\{\sup_{\lambda\in\mathbb{C}_+} \|\pi_-\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\pi_-\|, \qquad \sup_{\lambda\in\mathbb{C}_-} \|\pi_+\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\pi_+\|\right\} < \ \infty. \tag{3.4}$$

Then T is similar to a selfadjoint operator T_0. Moreover, if T is completely non-selfadjoint then T_0 has purely absolutely continuous spectrum.

The next result has originally been obtained in [55]. It is immediate from Theorem 3.2, other proofs can be found in [45, 40, 41].

Theorem 3.3 ([55]). *Let T be a quasi-selfadjoint extension of S and the spectrum $\sigma(T)$ is real, $\sigma(T) \subset \mathbb{R}$. If*

$$\sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-} \|\theta_T(\lambda)\| < \infty, \tag{3.5}$$

then T is similar to a selfadjoint operator T_0. Moreover, if T is completely non-selfadjoint then T_0 has purely absolutely continuous spectrum.

According to the B.S. Nagy and C. Foias result (see [57]) condition (3.5) is also necessary for a dissipative operator T to be similar to a selfadjoint operator.

To the best of our knowledge the most stronger sufficient condition of similarity of a non-dissipative operator to a selfadjoint one in terms of characteristic functions, is contained in Theorem 3.2. Some previous results in this direction can be found in [57], [55], [40], and [41] (see also references in [41]). We mention also recent publication [35] and [25].

Note that under the conditions of all mentioned results a completely nonselfadjoint part of T is similar to a selfadjoint operator $T_0 = T_0^*$ with absolutely continuous spectrum. In this connection we mention that Kapustin [25] found some sufficient conditions for an almost unitary operator T to be similar to an operator $U_{ac} \oplus T_s$ where U_{ac} is an absolutely continuous unitary operator and T_s is some singular almost unitary operator. Recall, that T is called an almost unitary operator, if $\sigma(T) \not\supset \mathbb{D}$ and (at least one of) non-unitary defects $I - T^*T$ and $I - TT^*$ are trace class operators.

Proposition 3.4. *Let a closed operator T on $\mathfrak{H}$ be similar to a selfadjoint operator $T_0 = T_0^*$, $VTV^{-1} = T_0$, and let $E_{T_0}(\cdot)$ be the spectral measure of T_0. Then*

(i) For any Borel subset $\delta \subset \mathbb{R}$ the subspace $\mathfrak{H}_T(\delta) := V^{-1}\mathfrak{H}_{T_0}(\delta)$, where $\mathfrak{H}_{T_0}(\delta) := E_{T_0}(\delta)\mathfrak{H}$ is a regularly and ultra-invariant invariant subspace for T;

(ii) The operator $T(\delta) := T\lceil\mathfrak{H}_T(\delta)$, $\operatorname{dom} T(\delta) = V^{-1}\operatorname{dom} T_0(\delta)$ is similar to the operator $T_0(\delta) := E_{T_0}(\delta)T$;

(iii) Suppose additionally that T is completely non-selfadjoint, $\sigma_p(T) = \emptyset$ and there exists a closed at most countable set $\{a_j\}_1^N \subset \mathbb{R}$, $N \le \infty$, such that for any domain

$$\mathcal{D} := \{\lambda \in \mathbb{C} : \ |\lambda| > \frac{1}{\varepsilon_\infty}\} \ \cup \ \bigcup_1^N \{\lambda \in \mathbb{C} : \ |\lambda - a_j| < \ \varepsilon_j\}$$

with sufficiently small ε_∞, ε_1, ε_2, $\ldots$, the following inequality holds

$$\sup_{\lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \setminus \mathcal{D}} \|\omega_\theta(\lambda)\| = \sup_{\lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \setminus \mathcal{D}} \|\mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| \ < \ \infty. \tag{3.6}$$

Then the spectrum of T_0 is purely absolutely continuous, that is T is similar to the selfadjoint operator T_0 with absolutely continuous spectrum.

Proof. (i) It is clear that $\mathfrak{H}_T(\delta) \in \operatorname{Lat} T$, that is $\mathfrak{H}_T(\delta)$ is invariant for T. Moreover, $\mathfrak{H}_T(\delta) \in \operatorname{Lat} T$ is regularly invariant, that is $(T-\lambda)^{-1}\mathfrak{H}_T(\delta) = \mathfrak{H}_T(\delta)$ since

$$E_{T_0}(\delta)\mathfrak{H} = (T_0 - \lambda)^{-1}E_{T_0}(\delta)\mathfrak{H} = V(T-\lambda)^{-1}V^{-1}E_{T_0}(\delta)\mathfrak{H} = V(T-\lambda)^{-1}\mathfrak{H}_T(\delta). \tag{3.7}$$

The last statement is a partial case of Proposition 5.1 from [57], part II.

(ii) It follows from the identity $VTV^{-1} = T_0$ that $V(T-\lambda)^{-1}V^{-1} = (T_0-\lambda)^{-1}$. Introducing block matrix representations of the operators V, $T(\delta)$ and $T_0(\delta)$ with respect to the orthogonal decompositions $\mathfrak{H} = \mathfrak{H}_T(\delta) \oplus \mathfrak{H}_T(\delta)^\perp = \mathfrak{H}_{T_0}(\delta) \oplus \mathfrak{H}_{T_0}(\mathbb{R} \setminus \delta)$ we rewrite the above identity in the block-matrix form

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \cdot \begin{pmatrix} \big(T(\delta) - \lambda\big)^{-1} & T_{12} \\ 0 & T_{22} \end{pmatrix} = \begin{pmatrix} \big(T_0(\delta) - \lambda\big)^{-1} & 0 \\ 0 & \big(T_0(\mathbb{R}\setminus\delta) - \lambda\big)^{-1} \end{pmatrix} \cdot \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{3.8}$$

where $V_{ij} = P_i V \lceil \mathfrak{H}_j$, $i,j \in \{1,2\}$, P_1 is the orthoprojection in $\mathfrak{H}$ onto $\mathfrak{H}_T(\delta)$ and $P_2 := I - P_1$. Hence $V_{11}\big(T(\delta) - \lambda\big)^{-1} = \big(T_0(\delta) - \lambda\big)^{-1}V_{11}$. To complete the proof it remains to note that $\operatorname{dom} V_{11} = \mathfrak{H}_T(\delta)$, $\operatorname{ran} V_{11} = \mathfrak{H}_{T_0}(\delta)$ and $\ker V_{11} = \{0\}$ by definition of V_{11}.

(iii) First we prove that the operator $T_2 := P_2 T\lceil\mathfrak{H}_T(\delta)^\perp$ is similar to the operator $T_0(\mathbb{R}\setminus\delta)$. Note that $T_2^* = T^*\lceil\mathfrak{H}_T(\delta)^\perp$ and

$$(V^{-1})^* T^* V^* = T_0 = T_0^*. \tag{3.9}$$

By statement (ii) the operator T_2^* is similar the operator $T_0(\mathbb{R}\setminus\delta)$ since $\mathfrak{H}_T(\delta)^\perp = V^*\mathfrak{H}_{T_0}(\mathbb{R}\setminus\delta) \in \operatorname{Lat} T^*$. Hence T_2 is similar to the operator $T_0(\mathbb{R}\setminus\delta) = T_0^*(\mathbb{R}\setminus\delta)$ too.

Now, let (a,b) be any component interval of the (open) set $\mathbb{R}\setminus\{a_j\}_1^N$ and $\delta = (a+\varepsilon, b-\varepsilon)$, $\varepsilon > 0$. It is clear that T is a coupling (see [5, 13, 14]) of $T_1 = T(\delta)$ and $T_2 = P_2 T\lceil\mathfrak{H}_T(\delta)^\perp$. Therefore $\theta_T(\cdot)$ admits a factorization (see [13, 14])

$$\theta_T(\lambda) = \theta_{T_1}(\lambda) \cdot \theta_{T_2}(\lambda) =: \theta_1(\lambda) \cdot \theta_2(\lambda), \qquad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{3.10}$$

where $\theta_j(\cdot) := \theta_{T_j}(\cdot)$ is the corresponding characteristic function of the operator $T_j, j \in \{1,2\}$. Since T_2 is similar to $T_0(\mathbb{R}\setminus\delta)$, then $\theta_2(\cdot) = \theta_{T_2}(\cdot)$ admits a holomorphic continuation through $(a+\varepsilon, b-\varepsilon)$.

It easily follows from (3.10) and the first $\mathcal{J}$-property of θ_{T_1} and θ_{T_2} (see (3.2)) that

$$\begin{aligned}\omega_\theta(\lambda) = \mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) &= \mathcal{J} - \theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda) + \theta_{T_1}(\lambda)\cdot(\mathcal{J} - \theta_{T_2}(\lambda)\mathcal{J}\theta_{T_2}^*(\lambda))\cdot\theta_{T_1}^*(\lambda) \\ &\geq \ \mathcal{J} - \theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda) \geq 0 \qquad \text{for} \qquad \lambda \in \mathbb{C}_+ \cup (a+\varepsilon, b-\varepsilon).\end{aligned} \tag{3.11}$$

In turn, it follows from (3.6) that $\omega_\theta(\cdot)$ is bounded in a small neighborhood G_δ^+ $(\subset \mathbb{C}_+)$ of $\delta = (a+\varepsilon, b-\varepsilon)$. Therefore (3.11) yields the estimate $\sup\limits_{\lambda\in G_\delta^+}\|\omega_{\theta_1}(\lambda)\| \leq \sup\limits_{\lambda\in G_\delta^+}\|\omega_{\theta_T}(\lambda)\| < \infty$.

On the other hand, $\theta_1(\lambda) = \theta_{T_1}(\lambda)$ is bounded at infinity since T_1 is bounded. Therefore $C_+ := \sup\limits_{\lambda\in\mathbb{C}_+}\|\omega_{\theta_1}(\lambda)\| < \infty$.

Similarly, starting with (3.10) and using the second $\mathcal{J}$-property (3.2) of θ_{T_1} and θ_{T_2} we get

$$\begin{aligned}\omega_\theta(\lambda) = \mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda) &= \mathcal{J} - \theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda) + \theta_{T_1}(\lambda)\cdot(\mathcal{J} - \theta_{T_2}(\lambda)\mathcal{J}\theta_{T_2}^*(\lambda))\cdot\theta_{T_1}^*(\lambda) \\ &\leq \ \mathcal{J} - \theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda) \leq 0 \qquad \text{for} \qquad \lambda \in \mathbb{C}_- \cup (a+\varepsilon, b-\varepsilon).\end{aligned} \tag{3.12}$$

By (3.6) $\omega_\theta(\cdot)$ is bounded in a small neighborhood G_δ^- $(\subset \mathbb{C}_-)$ of $\delta = (a+\varepsilon, b-\varepsilon)$ and due to (3.12) so is $\omega_{\theta_1}(\cdot)$. Since $\theta_1(\lambda)$ is bounded at infinity we have $C_- := \sup\limits_{\lambda\in\mathbb{C}_-}\|\omega_{\theta_1}(\lambda)\| < \infty$. Summing up we get

$$\sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-}\|\theta_{T_1}(\lambda)\mathcal{J}\theta_{T_1}^*(\lambda)\| \ < \ \infty. \tag{3.13}$$

Note that T_1 is completely nonselfadjoint because so is T. Since $T_1 = T(\delta)$ is completely nonselfadjoint and it is similar to the selfadjoint operator $T_0(\delta)$, then condition (3.13) imply absolute continuity of the operator $T_0(\delta)$ (see [41], Theorem 1.4). Since (a,b) is any component interval of $\mathbb{R}\setminus\{a_j\}_1^N$, $\delta = (a+\varepsilon, b-\varepsilon)$, and $\varepsilon > 0$ is arbitrary, then the singular spectrum $\sigma_s(T_0)$ of T_0 is supported on $\{a_j\}_1^N$, that is $\sigma_s(T_0) \subset \{a_j\}_1^N$. Thus, $\sigma_s(T_0)$ is at most countable, hence $\sigma_s(T_0) = \sigma_p(T_0)$. But according to our assumption $\sigma_p(T_0) = \emptyset$ and T_0 is purely absolutely continuous. $\square$

Corollary 3.5. *Let a closed operator T on $\mathfrak{H}$ be similar to a selfadjoint operator $T_0 = T_0^*$. Suppose additionally that T is completely non-selfadjoint, $\sigma_p(T) = \emptyset$ and there exists a closed at most countable set $\{a_j\}_1^N \subset \mathbb{R}$, $N \leq \infty$, such that for any domain*

$$\mathcal{D} := \{\lambda\in\mathbb{C}: \ |\lambda| > \frac{1}{\varepsilon_\infty}\} \ \cup \ \bigcup_1^N\{\lambda\in\mathbb{C}: \ |\lambda - a_j| < \ \varepsilon_j\}$$

with sufficiently small ε_∞, ε_1, ε_2, …, the following inequality holds

$$\sup_{\lambda\in\mathbb{C}_+\cup\mathbb{C}_-\setminus\mathcal{D}}\|\theta_T(\lambda)\| < \ \infty. \tag{3.14}$$

Then T_0 is purely absolutely continuous, that is T is similar to the selfadjoint operator T_0 with absolutely continuous spectrum.

Remark 3.1. It is shown in [41] that conditions (3.3) and (3.4) are equivalent to each other and even are equivalent to similar conditions obtaining by dropping the corresponding orthoprojections $\pi_\pm$. Note, however that in general condition (3.13) is weaker than each of the (equivalent) conditions (3.3) (3.4) and it is not sufficient for similarity to a selfadjoint operator (cf. [41]).

3.2 Characteristic functions and similarity of J-selfadjoint operators

In the case of J-selfadjoint operators conditions (3.3), (3.4) and (3.5) can be weaken. The following two results are immediate from Theorem 3.2 and Theorem 3.3 respectively.

Proposition 3.6. *Suppose additionally to the conditions of Theorem 3.2 that T is a J-selfadjoint operator. Assume also that $\sigma(T) \subset \mathbb{R}$ and at least one of the following four conditions is satisfied*

$$(i) \quad C_1 := \sup_{\lambda\in\mathbb{C}_+} \|\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\| < \infty, \qquad (ii) \quad C_2 := \sup_{\lambda\in\mathbb{C}_-} \|\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\| < \infty, \tag{3.15}$$

$$(iii) \quad C_3 := \sup_{\lambda\in\mathbb{C}_-} \|\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| < \infty, \qquad (iv) \quad C_4 := \sup_{\lambda\in\mathbb{C}_+} \|\theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| < \infty, \tag{3.16}$$

Then T is similar to a selfadjoint operator T_0. Moreover, if T is completely non-selfadjoint then T_0 has purely absolutely continuous spectrum.

Proof. If two operators T_1 and T_2 are unitarily equivalent, then any characteristic function $\theta_{T_1}(\cdot)$ of T_1 is at the same time the characteristic function of T_2.

We prove only that conditions (i) and (iii) are equivalent and $C_1 = C_3$. The equivalence $(ii) \iff (iv)$ and the equality $C_2 = C_4$ can be proved in just the same way.

Since T is J-selfadjoint it is unitarily equivalent to T^*, $T^* = JTJ^{-1}$. Hence $\theta_T(\lambda) = \theta_{T^*}(\lambda)$. On the other hand, it easily follows from (3.1), that

$$\theta_T^*(\overline{\lambda}) = \mathcal{J}\theta_{T^*}(\lambda)\mathcal{J} \ \big(= \mathcal{J}\theta_T(\lambda)^{-1}\mathcal{J}\big), \qquad \lambda \in \rho(T).$$

This relation yields

$$\theta_T^*(\overline{\lambda})\mathcal{J}\theta_T(\overline{\lambda}) = \theta_{T^*}(\lambda)\mathcal{J}\theta_{T^*}^*(\lambda) = \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda). \tag{3.17}$$

It follows that $C_1 = C_2$. To complete the proof it suffices to apply Theorem 3.2. □

Corollary 3.7. *Suppose additionally to the conditions of Theorem 3.3 that T is a J-selfadjoint operator. If $\sigma(T) \subset \mathbb{R}$ and*

$$\sup_{\lambda\in\mathbb{C}_+} \|\theta_T(\lambda)\| < \infty, \tag{3.18}$$

then T is similar to a selfadjoint operator T_0. Moreover, if T is completely non-selfadjoint then T_0 has purely absolutely continuous spectrum.

Remark 3.2. Note, that four conditions (i), (ii), (iii), (iv) in Proposition 3.6 are equaivalent. This statement is implied by combining identity (3.17) with Proposition 1.4 from [41].

In fact, it can proved using some reasonings from [41] based on the resolvent criterion (see below) that for J-selfadjoint operator T only "half" of either conditions (3.3) or conditions (3.4) is sufficient for T to be similar to a selfadjoint operator. Say, the condition $\sup_{\lambda\in\mathbb{C}_-} \|\pi_+\theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\pi_+\| < \infty$ is sufficient for T to be similar to a selfadjoint operator.

Next combining Proposition 3.4 with Proposition 3.6 we arrive at the following result showing that in the case of J-selfadjointness of the operator T condition (3.6) can also be weaken.

Proposition 3.8. *Let a closed J-selfadjoint operator T on $\mathfrak{H}$ be similar to a selfadjoint operator $T_0 = T_0^*$. Suppose additionally that T is completely non-selfadjoint, $\sigma_p(T) = \emptyset$ and there exists a closed at most countable set $\{a_j\}_1^N \subset \mathbb{R}$, $N \le \infty$, such that for any domain*

$$\mathcal{D} := \{\lambda \in \mathbb{C} : \ |\lambda| > \frac{1}{\varepsilon_\infty}\} \ \cup \ \bigcup_1^N \{\lambda \in \mathbb{C} : \ |\lambda - a_j| < \ \varepsilon_j\}$$

with sufficiently small ε_∞, ε_1, ε_2, ..., the following inequality holds

$$\sup_{\lambda \in \mathbb{C}_+ \setminus \mathcal{D}} \|\omega_\theta(\lambda)\| = \sup_{\lambda \in \mathbb{C}_+ \setminus \mathcal{D}} \|\mathcal{J} - \theta_T(\lambda)\mathcal{J}\theta_T^*(\lambda)\| < \ \infty. \tag{3.19}$$

Then T_0 is purely absolutely continuous, that is T is similar to the selfadjoint operator T_0 with absolutely continuous spectrum.

Proof. Since T is $\mathcal{J}$-selfadjoint, then combining condition (3.13) with identity (3.17) we get

$$\sup_{\lambda \in \mathbb{C}_- \setminus \mathcal{D}} \|\omega_{\theta^*}(\lambda)\| = \sup_{\lambda \in \mathbb{C}_- \setminus \mathcal{D}} \|\mathcal{J} - \theta_T^*(\lambda)\mathcal{J}\theta_T(\lambda)\| \ < \ \infty, \tag{3.20}$$

Following [41] it can easily be shown that both conditions (3.19) and (3.20) together yield condition (3.6). It remains to apply Proposition 3.4. □

Proposition 3.9. *Let $S := A_{\min}$ be a (minimal) symmetric operator defined by (2.3) and $A = JL$. Suppose that conditions of Proposition 2.5 are satisfied and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then*

(i) $B_I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} =: \mathcal{J}$ and the characteristic function $\theta_A(\cdot)$ of the operator A admits the following representation

$$\theta_A(\lambda) = \frac{1}{M_-(\lambda) - M_+(\lambda)} \begin{pmatrix} M_+(\lambda) + M_-(\lambda) & 2M_+(\lambda)M_-(\lambda) \\ 2 & M_+(\lambda) + M_-(\lambda) \end{pmatrix} \tag{3.21}$$

(ii) The corresponding $\mathcal{J}$-forms are

$$\begin{aligned} \omega_\theta(\lambda) &:= \mathcal{J} - \theta_A(\lambda)\mathcal{J}\theta_A^*(\lambda) \\ &= \mathcal{J} - \frac{1}{|M_+ - M_-|^2} \begin{pmatrix} 4 \cdot \operatorname{Im}\big(\overline{M_+ M_-} \cdot (M_+ + M_-)\big) & 4iM_+M_- - i|M_+ + M_-|^2 \\ i|M_+ + M_-|^2 - 4i\overline{M_+M_-} & 4 \cdot \operatorname{Im}\overline{(M_+ + M_-)} \end{pmatrix}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \omega_{\theta^*}(\lambda) &:= \mathcal{J} - \theta_A^*(\lambda)\mathcal{J}\theta_A(\lambda) \\ &= \mathcal{J} - \frac{1}{|M_+ - M_-|^2} \begin{pmatrix} 4 \cdot \operatorname{Im}\overline{(M_+ + M_-)} & 4iM_+M_- - i|M_+ + M_-|^2 \\ i|M_+ + M_-|^2 - 4i\overline{M_+M_-} & 4 \cdot \operatorname{Im}\big(\overline{M_+ M_-} \cdot (M_+ + M_-)\big) \end{pmatrix}. \end{aligned} \tag{3.23}$$

(iii) The determinant $\det \theta_A(\lambda)$ defined originally on $\rho(A^*)$, admits holomorphic continuation to the complex plane $\mathbb{C}$ and

$$\det \theta_A(\lambda) = 1, \qquad \lambda \in \mathbb{C}. \tag{3.24}$$

Combining Theorem 3.2 with Proposition 3.9 we arrive at the following statement.

Corollary 3.10. *Let $M_\pm$ be as above. Then the operator A is similar to a selfadjoint operator with absolutely continuous spectrum if the following two conditions hold*

$$(a)\quad \sup_{\lambda\in\mathbb{C}_+} \frac{\operatorname{Im}\big(M_+(\lambda)+M_-(\lambda)\big)+|M_+(\lambda)|^2\cdot\operatorname{Im}M_-(\lambda)+|M_-(\lambda)|^2\cdot\operatorname{Im}M_+(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2} \ <\ \infty, \tag{3.25}$$

$$(b)\quad \sup_{\lambda\in\mathbb{C}_+} \frac{\operatorname{Im}M_+(\lambda)\cdot\operatorname{Im}M_-(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2} \ <\ \infty. \tag{3.26}$$

Proof. Note that $\pi_\pm := (I\pm\mathcal{J})/2 = \frac{1}{2}\begin{pmatrix}1 & \mp i\\ \pm i & 1\end{pmatrix}$. Setting for brevity $\begin{pmatrix}a & b\\ c & d\end{pmatrix} := \mathcal{J}-\omega_{\theta^*}(\lambda)$ and noting that $\mathcal{J}-\omega_\theta(\lambda) = \begin{pmatrix}d & b\\ c & a\end{pmatrix}$ we easily get

$$\pi_+\omega_{\theta^*}(\lambda)\pi_+ = \pi_+ - \frac{1}{4}\begin{pmatrix}k_+ & -ik_+\\ ik_+ & k_+\end{pmatrix}, \qquad \pi_-\omega_\theta(\lambda)\pi_- = -\pi_- - \frac{1}{4}\begin{pmatrix}k_- & ik_-\\ -ik_- & k_-\end{pmatrix}, \tag{3.27}$$

where $k_+ = a-ic+ib+d$ and $k_- = a+ic-ib+d$. Hence both k_+ and k_- are bounded in $\mathbb{C}_+$ if and only if so are $a+d = k_++k_-$ and $b-c = i(k_--k_+)$. Note that

$$\frac{c-b}{2i} = \frac{|M_+(\lambda)+M_-(\lambda)|^2-4Re\big(M_+(\lambda)\cdot M_-(\lambda)\big)}{|M_+(\lambda)-M_-(\lambda)|^2} = 1+\frac{8\operatorname{Im}M_+(\lambda)\cdot\operatorname{Im}M_-(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2}\ ,$$

$$\operatorname{Im}\big(M_+(\lambda)\cdot M_-(\lambda)\cdot\overline{(M_+(\lambda)+M_-(\lambda))}\big) = |M_+(\lambda)|^2\cdot\operatorname{Im}M_-(\lambda)+|M_-(\lambda)|^2\cdot\operatorname{Im}M_+(\lambda).$$

To complete the proof it remains to apply Theorem 3.2. $\square$

Remark 3.3. (i) A weaker sufficient condition of similarity is implied by Theorem 3.3. Namely, combining Theorem 3.3 with formula (3.21) we conclude that the condition

$$\max\left\{\sup_{\lambda\in\mathbb{C}_+}\frac{|M_++M_-|}{|M_--M_+|}\ ,\qquad \sup_{\lambda\in\mathbb{C}_+}\frac{1}{|M_--M_+|}\ ,\qquad \sup_{\lambda\in\mathbb{C}_+}\frac{|M_+M_-|}{|M_--M_+|}\right\}\ <\ \infty \tag{3.28}$$

is sufficient for the operator A to be similar to a selfadjoint operator with absolutely continuous spectrum.

(ii) A counter part of identity (3.24) for a discrete part A_{disc} of the operator A, $\det\theta_{A_{disc}}(\lambda)=1$, is immediate from symmetry of its spectrum (see Proposition 2.5(vi)). However, identity (3.24) is not predictable for operators with absolutely continuous spectrum. In the latter case $\theta_A(\cdot)$ is j-outer function while $\det\theta_A(\lambda)=1$.

Alongside the operator A we consider its "dissipative and accumulative parts". More precisely, we consider extensions $A_\pm$ of $S=A_{\min}$ determined by

$$\operatorname{dom}(A_\pm) :=\ \{y\in\operatorname{dom}((S^*) : 2y'(+0) = y'(-0)\pm iy(+0),\ 2y(-0) = y(+0)\mp iy'(-0)\}\,. \tag{3.29}$$

Proposition 3.11. *Let $S := A_{\min}$ be a (minimal) symmetric operator defined by* (2.3) *and let $M_\pm(\cdot)$ be defined by* (2.6). *Let also $\Pi = \{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$ be a boundary triplet defined by* (2.24). *Then*

(i) *The operators $A_\pm$ defined by (2.2) are quasi-selfadjoint extensions of S and they are determined by*

$$A_\pm = S^*|\operatorname{dom} A_\pm, \qquad \operatorname{dom} A_\pm = \ker(\Gamma_1 - B_\pm\Gamma_0), \quad \textit{and} \quad B_\pm := \pi_\pm B = \frac{1}{2}\begin{pmatrix} \pm i & 1 \\ -1 & \pm i \end{pmatrix}, \tag{3.30}$$

that is $A_\pm = S_{B_\pm}$, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{J} = -iB = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \textit{and} \quad \pi_\pm := (I \pm \mathcal{J})/2 = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}. \tag{3.31}$$

(ii) *Some of the characteristic functions of the operators $A_\pm$ are*

$$\theta_{A_\pm}(\lambda) = I - \frac{1 - M_+(\lambda)M_-(\lambda)}{\Delta_\pm(\lambda)}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}, \tag{3.32}$$

where $\Delta_\pm(\lambda) := 1 - M_+(\lambda)M_-(\lambda) \mp 2iM_-(\lambda)$.

(iii) *The operator A_+ (resp A_-) is similar to a selfadjoint operator if and only if*

$$\inf_{\lambda\in\mathbb{C}_-} |1 - i\Phi(\lambda)| =: \varepsilon \ > \ 0, \tag{3.33}$$

where

$$\Phi(\cdot) := 2(M_-^{-1}(\cdot) - M_+(\cdot))^{-1} \in (R).$$

Proof. (i) This statement is obvious.

(ii) This statement is implied by combining formula (3.1) with (3.30) and (3.31).

(iii) First we note that by (3.33)

$$\sup_{\lambda\in\mathbb{C}_-}\left|\frac{1 - M_+(\lambda)M_-(\lambda)}{\Delta_\pm(\lambda)}\right| = \left|\frac{1}{1 - i\Phi(\lambda)}\right| = \frac{1}{\varepsilon} \ < \ \infty.$$

Therefore it follows from (3.32) that condition (3.33) is equivalent to the boundedness of the characteristic function $\theta_{A_+}(\cdot)$ in $\mathbb{C}_-$..

Now the result is immediate from the B.S. Nagy and Foias [57] criterion. □

3.3 Resolvent criterion

It turns out, that in general conditions (3.5), (3.3), (3.4) are not satisfied for the operators of type (2.2), though such operators may be similar to a selfadjoint operator (see [41]).

Our approach is based on the resolvent similarity criterion obtained in [45] and [40] (under an additional assumption this criterion was obtained in [6], another proof has also been obtained in [20]).

Theorem 3.12 ([45, 40]). *A closed operator T on a Hilbert space $\mathfrak{H}$ is similar to a selfadjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and for all $f \in \mathfrak{H}$ the inequalities*

$$\sup_{\varepsilon>0}\ \varepsilon\cdot\int_{\mathbb{R}} \|\mathcal{R}_T(\eta + i\varepsilon)f\|^2\, d\eta \le K_1\,\|f\|^2, \qquad \sup_{\varepsilon>0}\ \varepsilon\cdot\int_{\mathbb{R}} \|\mathcal{R}_{T^*}(\eta + i\varepsilon)f\|^2\, d\eta \le K_{1*}\,\|f\|^2, \tag{3.34}$$

hold with constants K_1 and K_{1*} independent of f.

The following proposition is immediate from Theorem 3.12.

Proposition 3.13. *A J-selfadjoint operator T on a Hilbert space $\mathfrak{H}$ is similar to a selfadjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and the following inequality holds*

$$\sup_{\varepsilon>0} \varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_T(\eta + i\varepsilon)f\|^2 \, d\eta \leq K_1 \|f\|^2, \qquad f \in \mathfrak{H}, \tag{3.35}$$

with a constant K_1 independent of f.

Proof. If T is a J-selfadjoint operator, then $T^* = JTJ$ and the second inequality in (3.34) is equivalent to the first one. $\square$

In the case of a bounded operator T we can slightly clarify Theorem 3.12 in the following way.

Corollary 3.14. *Let $T = T_1 + iT_2$ where $T_1 = T_1^*$ and $T_2 = T_2^* \in [\mathfrak{H}]$. Then T is similar to a selfadjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and for all $f \in \mathfrak{H}$ the inequalities*

$$\sup_{0<\varepsilon<2\|T_2\|} \varepsilon \int_{\mathbb{R}} \|\mathcal{R}_T(\eta + i\varepsilon)f\|^2 \, d\eta \leq K_1 \|f\|^2, \qquad \sup_{0<\varepsilon<2\|T_2\|} \varepsilon \int_{\mathbb{R}} \|\mathcal{R}_{T^*}(\eta + i\varepsilon)f\|^2 \, d\eta \leq K_{1*} \|f\|^2, \tag{3.36}$$

hold with constants K_1 and K_{1*} independent of $f \in \mathfrak{H}$.

In particular, a bounded operator T on $\mathfrak{H}$ with $\sigma(T) \subset \mathbb{R}$ is similar to a selfadjoint operator if and only if inequalities (3.36) *are valid with* $2\|T_2\|$ *replaced for any* $\varepsilon_0 > 0$.

Proof. (i) It is clear that

$$(T - z)^{-1} = (T_1 - z)^{-1} - (T_1 - z)^{-1} \cdot T_2 \cdot (T - z)^{-1}, \qquad z \in \mathbb{C}_+.$$

It follows that

$$\begin{aligned}
\|(T - z)^{-1}f\|^2 &\leq 2\|(T - z)^{-1}f\|^2 + 2\|(T_1 - z)^{-1} \cdot T_2 \cdot (T - z)^{-1}f\|^2 \\
&\leq \|(T_1 - z)^{-1}f\|^2 + \frac{2\|T_2\|^2}{|\operatorname{Im} z|}\|(T - z)^{-1}f\|^2 \qquad z \in \mathbb{C}_+, \quad f \in \mathfrak{H}.
\end{aligned}$$

In turn, this inequality yields $\|(T - z)^{-1}f\| \leq 2\|(T_1 - z)^{-1}f\|$ for $\operatorname{Im} z > 2\|T_2\|$. Hence

It is known that $\|(T - z)^{-1}\| \leq (|z| - \|T\|)^{-1}$ for $|z| > \|T\|$. Hence

$$\sup_{\varepsilon \geq 2\|T_2\|} \varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_T(\eta + i\varepsilon)f\|^2 \, d\eta \leq 4\varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_{T_1}(\eta + i\varepsilon)f\|^2 \, d\eta = 4\pi \|f\|^2. \tag{3.37}$$

Combining this inequality with the first of inequalities (3.34) we arrive at the first of inequalities (3.36). The second one can be proved similarly. $\square$

Remark 3.4. If T is a closed unbounded operator, then conditions (3.34) and (3.36) are not equivalent, in general. In fact, there exists an operator T such that:

(i) $\sigma(T) \subset \mathbb{R}$;

(ii) conditions (3.36) are fulfilled;

(iii) conditions (3.34) do not hold and, consequently, T is not similar to a self-adjoint operator.

Setting

$$\begin{cases} w(x) = 1, & x \in (-\infty, -1) \cup (1, +\infty), \\ w(x) = -1, & x \in (-1, 1), \end{cases}$$

consider the operator $D_w = -i\frac{1}{w(x)}\frac{d}{dx}$ in $L^2(\mathbb{R})$. It was shown in [30] that the operator D_w has the properties (i) and (iii). It is not difficult to check that conditions (3.36) are fulfilled for D_w.

4 Eigenvalues and their multiplicities

In [31, 32] the functional model for J-selfadjoint quasiselfadjoint operator was given. The model is based on classical Sturm-Liouville spectral theory and the functional model for a symmetric operator given in Section 2.5.

Let $\Sigma_\pm$ be the spectral functions of $A_0^\pm$ (see (2.8)). It follows from (2.10) that they satisfy (2.28). Let $C_\pm := \displaystyle\int_{\mathbb{R}} \frac{t}{1+t^2} d\Sigma_\pm$. We denote $\widehat{\Gamma}_0^\pm := \Gamma_0^{\Sigma_\pm}$, $\widehat{\Gamma}_1^\pm := \Gamma_1^{\Sigma_\pm, C_\pm}$. From the definition of $C_\pm$ and (2.29), we get

$$\widehat{\Gamma}_1^\pm f = C_\pm \widehat{\Gamma}_0^\pm f + \int_{\mathbb{R}} \left(f(t) - \frac{t\,\widehat{\Gamma}_0^\pm f}{t^2+1} \right) d\Sigma_\pm(t) = \int_{\mathbb{R}} f(t) d\Sigma_\pm(t)$$

for $f \in \operatorname{dom}(\widehat{T}_{\Sigma_\pm})$. Consider the operator $\widehat{A}$ in $L^2(d\Sigma_+) \oplus L^2(d\Sigma_-)$ defined by

$$\begin{gathered} \widehat{A} = \widehat{T}^*_{\Sigma_+} \oplus \widehat{T}^*_{\Sigma_-} \restriction \operatorname{dom}(\widehat{A}), \\ \operatorname{dom}(\widehat{A}) = \{\, f = f_+ + f_- \ : \ f_\pm \in \operatorname{dom}(\widehat{T}^*_{\Sigma_\pm}),\ \widehat{\Gamma}_0^+ f_+ = \widehat{\Gamma}_0^- f_-,\ \widehat{\Gamma}_1^+ f_+ = \widehat{\Gamma}_1^- f_- \,\} \end{gathered} \tag{4.1}$$

(for the definition of $\widehat{T}_{\Sigma_\pm}$ see Section 2.5).

Proposition 4.1 ([31, 32]). *The operator A of type* (2.2) *is unitary equivalent to the operator $\widehat{A}$. Moreover,*

$$(\mathcal{F}_- \oplus \mathcal{F}_+) A (\mathcal{F}_-^{-1} \oplus \mathcal{F}_+^{-1}) = \widehat{A}\ . \tag{4.2}$$

Note that we can write the Weyl functions of A in the form

$$M_\pm(\lambda) = M_{\Sigma_\pm, C_\pm}(\lambda)\ , \quad \lambda \in \mathbb{C} \setminus \operatorname{supp} d\Sigma_\pm$$

(see (2.30) for the definition of $M_{\Sigma_\pm, C_\pm}$).

Now we classify eigenvalues of $\widehat{T}^*_\Sigma$. Let us introduce the following mutually disjoint sets:

$$\mathfrak{A}_0(\Sigma) = \left\{ \lambda \in \sigma_c(Q_\Sigma) \ : \ \int_{\mathbb{R}} |t-\lambda|^{-2} d\Sigma(t) = \infty \right\},$$

$$\mathfrak{A}_r(\Sigma) = \left\{ \lambda \notin \sigma_p(Q) \ : \ \int_{\mathbb{R}} |t-\lambda|^{-2} d\Sigma(t) < \infty \right\}, \qquad \mathfrak{A}_p(\Sigma) = \sigma_p(Q_\Sigma).$$

Observe that $\mathbb{C} = \mathfrak{A}_0(\Sigma) \cup \mathfrak{A}_r(\Sigma) \cup \mathfrak{A}_p(\Sigma)$ and

$$\mathfrak{A}_0(\Sigma) = \{\lambda \in \mathbb{C} \ : \ \ker(A_\Sigma^* - \lambda) = \{0\} \ \},$$
$$\mathfrak{A}_r(\Sigma) = \left\{\lambda \in \mathbb{C} : \ \ker(A_\Sigma^* - \lambda) = \{c(t-\lambda)^{-1}, \ c \in \mathbb{C}\}\right\}, \tag{4.3}$$
$$\mathfrak{A}_p(\Sigma) = \left\{\lambda \in \mathbb{C} \ : \ \ker(A_\Sigma^* - \lambda) = \{c\chi_{\{\lambda\}}(t), \ c \in \mathbb{C}\}\right\}. \tag{4.4}$$

The following theorem gives a description of the point spectrum of $\widehat{A}$.

Theorem 4.2 ([31, 32]). *Let $\widehat{A}$ be given by (4.1).*

1) *If $\lambda \in \mathfrak{A}_0(\Sigma_+) \cup \mathfrak{A}_0(\Sigma_-)$, then $\lambda \notin \sigma_p(\widehat{A})$.*

2) *If $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_p(\Sigma_-)$, then*

(i) *λ is an eigenvalue of $\widehat{A}$; the geometric multiplicity of λ equals 1;*

(ii) *the eigenvalue λ is simple (i.e., the algebraic and geometric multiplicities are equal one) iff at least one of the following conditions is not fulfilled:*

$$\Sigma_-(\lambda+0) - \Sigma_-(\lambda-0) = \Sigma_+(\lambda+0) - \Sigma_+(\lambda-0), \tag{4.5}$$
$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^2} \, d\Sigma_+(t) < \infty, \tag{4.6}$$
$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^2} \, d\Sigma_-(t) < \infty; \tag{4.7}$$

(iii) *if conditions (4.5), (4.6) and (4.7) hold true, then the algebraic multiplicity of λ equals the greatest number k ($2 \le k \le \infty$) such that the following conditions*

$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^{2j}} \, d\Sigma_-(t) < \infty, \qquad \int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{|t-\lambda|^{2j}} \, d\Sigma_+(t) < \infty, \tag{4.8}$$
$$\int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{(t-\lambda)^{j-1}} \, d\Sigma_-(t) = \int_{\mathbb{R}\setminus\{\lambda\}} \frac{1}{(t-\lambda)^{j-1}} \, d\Sigma_+(t), \tag{4.9}$$

are fulfilled for all $j \in \mathbb{N} \cap [2, k-1]$.

3) *Assume that $\lambda \in \mathfrak{A}_r(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_+)$. Then $\lambda \in \sigma_p(\widehat{A})$ iff*

$$\int_{\mathbb{R}} \frac{1}{t-\lambda} d\Sigma_+(t) = \int_{\mathbb{R}} \frac{1}{t-\lambda} d\Sigma_-(t) \ . \tag{4.10}$$

If (4.10) holds true, then the geometric multiplicity of λ is one and the algebraic multiplicity is the greatest number k ($1 \le k \le \infty$) such that the following conditions

$$\int_{\mathbb{R}} \frac{1}{|t-\lambda|^{2j}} \, d\Sigma_-(t) < \infty, \qquad \int_{\mathbb{R}} \frac{1}{|t-\lambda|^{2j}} \, d\Sigma_+(t) < \infty, \tag{4.11}$$
$$\int_{\mathbb{R}} \frac{1}{(t-\lambda)^j} \, d\Sigma_-(t) = \int_{\mathbb{R}} \frac{1}{(t-\lambda)^j} \, d\Sigma_+(t) \tag{4.12}$$

are fulfilled for all $j \in \mathbb{N} \cap [1, k]$.

4) *If* $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$ *or* $\lambda \in \mathfrak{A}_p(\Sigma_-) \cap \mathfrak{A}_r(\Sigma_+)$, *then* $\lambda \notin \sigma_p(\widehat{A})$.

It follows from Theorem 4.2 (as well as from Proposition 2.5) that

$$\big\{\lambda \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-}) \ : \ M_+(\lambda) = M_-(\lambda)\big\} = \sigma(\widehat{A}) \cap \rho(Q_{\Sigma_+} \oplus Q_{\Sigma_-}) \subset \sigma_p(\widehat{A}). \tag{4.13}$$

It is easy to see that (4.13) and Theorem 4.2 yield the following description of the essential and discrete spectra.

Proposition 4.3 ([31, 32]). 1) $\sigma_{ess}(\widehat{A}) = \sigma_{ess}(Q_{\Sigma_+}) \cup \sigma_{ess}(Q_{\Sigma_-})$;

2) $\sigma_{disc}(\widehat{A}) = \big(\sigma_{disc}(Q_{\Sigma_+}) \cap \sigma_{disc}(Q_{\Sigma_-})\big) \ \cup \ \{\lambda \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-}) \ : \ M_+(\lambda) = M_-(\lambda)\}$;

3) *the geometric multiplicity equals 1 for all eigenvalues of* $\widehat{A}$;

4) *if* $\lambda_0 \in \big(\sigma_{disc}(Q_{\Sigma_+}) \cap \sigma_{disc}(Q_{\Sigma_-})\big)$, *then the algebraic multiplicity of* λ_0 *is equal to the multiplicity of* λ_0 *as a zero of the holomorphic function* $\dfrac{1}{M_+(\lambda)} - \dfrac{1}{M_-(\lambda)}$;

5) *if* $\lambda_0 \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-})$ *then the algebraic multiplicity of* λ_0 *is equal to the multiplicity of* λ_0 *as zero of the holomorphic function* $M_+(\lambda) - M_-(\lambda)$.

Proposition 4.4. *Let* A *be the operator defined by* (2.2) *and* $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. *Then*

(i) $\rho(A) \neq \emptyset$ *and* $\lambda_0 \in \rho(A) \cap \mathbb{C}_\pm$ *if and only if* $M_+(\lambda_0) \neq M_-(\lambda_0)$.

(ii) *The resolvent of* A *has the following form*

$$\mathcal{R}_A(\lambda) f(\cdot) = \mathcal{R}_{A_0^- \oplus A_0^+}(\lambda) f(\cdot) + G^-(\lambda)\psi_-(\cdot, \lambda) + G^+(\lambda)\psi_+(\cdot, \lambda), \tag{4.14}$$

$$G^-(\lambda) = G^+(\lambda) = \frac{1}{M_+(\lambda) - M_-(\lambda)} \int_{\mathbb{R}} \frac{g^-(t) d\Sigma_-(t) - g^+(t) d\Sigma_+(t)}{t - \lambda}, \tag{4.15}$$

where $g^\pm(t) = (\mathcal{F}_\pm f_\pm)(t)$, $f_\pm := P_\pm f \in L^2(\mathbb{R}_\pm)$, *and* $P_\pm$ *is the orthoprojection in* $L^2(\mathbb{R})$ *onto* $L^2(\mathbb{R}_\pm)$.

Proof. (i) This statement has already been proved in Proposition 2.5.

(ii) Let now $\lambda \in \rho(A)$ and $y(\cdot, \lambda) = (A - \lambda I)^{-1} f(\cdot)$. It means that $y \in \operatorname{dom}(A^*_{min})$ and y is a solution of the equation

$$(\operatorname{sgn} x)(-y''(x) + q(x)y(x)) - \lambda y(x) = f(x) \tag{4.16}$$

subject to "glue" boundary conditions

$$y(-0) = y(+0), \qquad y'(-0) = y'(+0) \ . \tag{4.17}$$

Hence,

$$y(x, \lambda) = (\mathcal{R}_{A_0^- \oplus A_0^+}(\lambda) f)(x) + G^-(\lambda)\psi_-(x, \lambda) + G^+(\lambda)\psi_+(x, \lambda),$$

where $G^\pm(\lambda)$ are the scalar functions. It is clear that

$$y(\pm 0, \lambda) = (\mathcal{R}_{A_0^\pm}(\lambda) f_\pm)(\pm 0) + G^\pm(\lambda)\psi_\pm(0, \lambda).$$

By (2.7), we get

$$\psi_\pm(0,\lambda) = M_\pm(\lambda), \qquad \frac{d}{dx}\psi_\pm(0,\lambda) = -1. \tag{4.18}$$

Resolvent representation (2.13) yields

$$(\mathcal{R}_{A_0^\pm}(\lambda)f_\pm)(\pm 0) = \int_{\mathbb{R}} \frac{g^\pm(t)d\Sigma_\pm(t)}{t-\lambda}.$$

It follows from $\mathcal{R}_{A_0^\pm}(\lambda)f_\pm \in D(A_2^\pm)$ and (2.11) that $\frac{d}{dx}(\mathcal{R}_{A_0^\pm}(\lambda)f_\pm)_{x=\pm 0} = 0$. Taking into account (4.18), we see that conditions (4.17) take the form

$$\begin{cases} \displaystyle\int_{\mathbb{R}} \frac{g^-(t)d\Sigma_-(t)}{t-\lambda} + G^-(\lambda)M_-(\lambda) = \int_{\mathbb{R}} \frac{g^+(t)d\Sigma_+(t)}{t-\lambda} + G^+(\lambda)M_+(\lambda) \\ G^-(\lambda) = G^+(\lambda) \end{cases}.$$

Since $M_+(\lambda) \neq M_-(\lambda)$, problem (4.16)-(4.17) has the unique solution $y \in \operatorname{dom}(A^*_{min})$ and it admits a representation (4.14)-(4.15). $\square$

Next we clarify Proposition 4.4 in the case of J-nonnegative operator A, i.e., if $L \geq 0$.

Proposition 4.5. *If the operator $L = -d^2/dx^2 + q(x)$ is nonnegative, then the spectrum of the operator $A = JL$ is real.*

Proof. Since $L \geq 0$ we have $A^+_{\min} = L^+_{\min} \geq 0$ and $A^-_{\min} = -L^+_{\min} \leq 0$. It is known that the Friedrichs extension $L_F^\pm$ of $L^\pm_{\min}$ is generated by the Dirichlet boundary value problem, that is

$$L_F^\pm = (L^\pm_{\min})^* \lceil \operatorname{dom}(L_F^\pm), \quad \operatorname{dom}(L_F^\pm = \{f \in \operatorname{dom}(L^\pm_{\min})^* : f(0) = 0\}. \tag{4.19}$$

Setting $\Gamma_0^\pm f = f(\pm 0)$ and $\Gamma_1^\pm f = \pm f'(\pm 0)$ we obtain a boundary triplet $\Pi_\pm = \{\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\}$ for $(L^\pm_{\min})^*$ such that $\ker \Gamma_0^\pm = \operatorname{dom}(L_F^\pm)$. Therefore the corresponding Weyl function $m_F^\pm$ belongs to the Krein-Stielties class S^- (see [14]). Hence, it admits the following integral representation (see [24]).

$$m_F^\pm(\lambda) = C_\pm + \lambda \int_0^\infty \frac{d\sigma_\pm(t)}{t-\lambda}, \qquad \int_0^\infty \frac{d\sigma_\pm(t)}{1+t} < \infty, \tag{4.20}$$

with $C_\pm \leq 0$. On the other hand, it follows from definitions that

$$-M_+^{-1}(\lambda) = -m_+^{-1}(\lambda) = m_F^+(\lambda), \qquad M_-^{-1}(\lambda) = -m_-^{-1}(\lambda) = m_F^-(-\lambda) \tag{4.21}$$

Combining (4.20) and (4.21) we get

$$\begin{aligned} M_-^{-1}(\lambda) - M_+^{-1}(\lambda) &= m_F^-(-\lambda) + m_F^+(\lambda) \\ &= \lambda\Big[\frac{C_- + C_+}{\lambda} + \int_0^\infty \frac{d\sigma_+(t)}{t-\lambda} - \int_0^\infty \frac{d\sigma_-(t)}{t+\lambda}\Big] =: \lambda\widetilde{M}(\lambda), \end{aligned} \tag{4.22}$$

where $\widetilde{M}(\cdot) \in (R)$ since $C_\pm \leq 0$. To complete the proof it remains to note that

$$M_+(\lambda) - M_-(\lambda) = M_+(\lambda)\cdot[M_-^{-1}(\lambda) - M_+^{-1}(\lambda)]\cdot M_-(\lambda) = M_+(\lambda)\cdot\lambda\widetilde{M}(\lambda)\cdot M_-(\lambda) \neq 0 \tag{4.23}$$

for $\lambda \in \mathbb{C}_\pm$, since $M_\pm, \widetilde{M} \in (R)$. $\square$

Remark 4.1. (i) Statement (i) of Proposition 4.4 is implied by (4.13). However, we presented an elementary proof based on Proposition 2.4.

(ii) Note that Proposition 4.5 follows immediately from Proposition 4.4 and Proposition 2.2. However, we presented another proof that is in a spirit of our paper and demonstrates applicability of Weyl function technic. Note also that in turn, Proposition 2.2 can be proved by using Weyl function technic similarly to the proof of Proposition 4.5.

5 Similarity conditions for the operator A. General case.

5.1 Similarity criterion in terms of Weyl functions.

In the sequel we write $\lambda = \eta + i\varepsilon$, that is $\eta = \operatorname{Re}\lambda$, $\varepsilon = \operatorname{Im}\lambda$.

Combining Proposition 3.13 and Proposition 4.4, we arrive at the following criterion.

Theorem 5.1. *The operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ is similar to a selfadjoint operator if and only if for all $\varepsilon > 0$ and $g^\pm \in L^2(\mathbb{R}, d\Sigma_\pm)$ the following inequalities hold:*

$$\int\limits_{\eta\in\mathbb{R}} \frac{\operatorname{Im} M_\pm(\eta+i\varepsilon)}{|M_+(\eta+i\varepsilon) - M_-(\eta+i\varepsilon)|^2} \left| \int_{\mathbb{R}} \frac{g^-(t)d\Sigma_-(t)}{t-(\eta+i\varepsilon)} \right|^2 d\eta \le K^- \|g^-\|^2_{L^2(d\Sigma_-)}, \tag{5.1}$$

$$\int\limits_{\eta\in\mathbb{R}} \frac{\operatorname{Im} M_\pm(\eta+i\varepsilon)}{|M_+(\eta+i\varepsilon) - M_-(\eta+i\varepsilon)|^2} \left| \int_{\mathbb{R}} \frac{g^+(t)d\Sigma_+(t)}{t-(\eta+i\varepsilon)} \right|^2 d\eta \le K^+ \|g^+\|^2_{L^2(d\Sigma_+)}, \tag{5.2}$$

where $K^\pm$ are constants independent of ε and $g^\pm$.

Proof. It is known (see [54]) that for any selfadjoint $B = B^*$ with resolution of identity E_t^B the following identity holds

$$\varepsilon \cdot \int\limits_{\eta\in\mathbb{R}} \|\mathcal{R}_B(\eta+i\varepsilon)f\|^2 \, d\eta = \pi\|f\|^2, \qquad \varepsilon > 0, \quad f \in \mathfrak{H}. \tag{5.3}$$

It follows from (4.14) that

$$\|\mathcal{R}_A(\lambda)f\|^2 - 2\|\mathcal{R}_{A_0^-\oplus A_0^+}f\|^2 \le 2\|G^-(\lambda)\psi_-(\lambda) + G^+(\lambda)\psi_+(\lambda)\|^2$$
$$\le 4\|\mathcal{R}_A(\lambda)f\|^2 + 4\|\mathcal{R}_{A_0^-\oplus A_0^+}f\|^2.$$

On the other hand, it follows from (5.3) with B replaced by $A_0^- \oplus A_0^+$ that

$$\begin{aligned}
&\frac{\varepsilon}{2}\int\limits_{\eta\in\mathbb{R}} \|\mathcal{R}_A(\eta+i\varepsilon)f\|^2 d\eta - \pi\|f\|^2 \le \\
&\le \varepsilon \int\limits_{\eta\in\mathbb{R}} \|G^-(\eta+i\varepsilon)\psi_-(\eta+i\varepsilon) + G^+(\eta+i\varepsilon)\psi_+(\eta+i\varepsilon)\|^2 d\eta \le \\
&\le 2\varepsilon \int\limits_{\eta\in\mathbb{R}} \|\mathcal{R}_A(\eta+i\varepsilon)f\|^2 d\eta + 2\pi\|f\|^2.
\end{aligned} \tag{5.4}$$

Since $\psi_\pm \in L^2(\mathbb{R}_\pm, dx)$ and $\|\psi_\pm(\cdot,\lambda)\|^2_{L^2(\mathbb{R}_\pm)} = \operatorname{Im} M_\pm(\lambda)/\operatorname{Im}\lambda$ (see [39]), we have

$$\begin{aligned}
&\|G^-(\lambda)\psi_-(\cdot,\lambda) + G^+(\lambda)\psi_+(\cdot,\lambda)\|^2 = \\
&= |G^-(\lambda)|^2\|\psi_-(\cdot,\lambda)\|^2 + |G^+(\lambda)|^2\|\psi_+(\cdot,\lambda)\|^2 = \\
&= \frac{1}{|M_+(\lambda) - M_-(\lambda)|^2} \left| \int\limits_{\mathbb{R}} \frac{g^-(t)d\Sigma_-(t) - g^+(t)d\Sigma_+(t)}{t-\lambda} \right|^2 \frac{\operatorname{Im} M_+(\lambda) + \operatorname{Im} M_-(\lambda)}{\operatorname{Im}\lambda}.
\end{aligned}$$

Combining this relation with (5.4) one concludes that (3.35) is equivalent to the following condition

$$\int\limits_{\eta\in\mathbb{R}} \frac{\operatorname{Im} M_+(\eta+i\varepsilon)+\operatorname{Im} M_-(\eta+i\varepsilon)}{|M_+(\eta+i\varepsilon)-M_-(\eta+i\varepsilon)|^2}\left|\int_R \frac{g^-(t)d\Sigma_-(t)-g^+(t)d\Sigma_+(t)}{t-(\eta+i\varepsilon)}\right|^2 d\eta \le C_1\|f\|^2, \qquad (5.5)$$

where C_1 is a constant independent of f and ε.

By definition, $\|g^\pm\|_{L^2(d\Sigma_\pm)} = \|f_\pm\|_{L^2(\mathbb{R}_\pm)}$, where $f_\pm = P_\pm f$. Thus, condition (5.5) holds iff both (5.2) and (5.1) are satisfied. ☐

5.2 Necessary conditions of similarity in terms of the Weyl functions and Hilbert transforms.

Let $\Sigma_\pm = \Sigma_{ac\pm} + \Sigma_{s\pm} = \Sigma_{ac\pm} + \Sigma_{sc\pm} + \Sigma_{d\pm}$ be the Lebesgue decomposition of the measure $\Sigma_\pm$ into a sum of absolutely continuous, singular continuous, and pure point measures (see, for example, [53]).

Denote by $S'_{ac}(\Sigma_\pm)$ and $S'_s(\Sigma_\pm)$ mutually disjoint (not necessarily topological) supports of measures $\Sigma_{ac\pm}$ and $\Sigma_{s\pm}$, respectively.

Note that for almost all $t\in\mathbb{R}$ the nontangential limit

$$\lim_{\substack{\lambda\to t\\ \sphericalangle}} M_\pm(\lambda) =: M_\pm(t)$$

exists (see [18]). Since $M_+(\lambda)\not\equiv M_-(\lambda)$ on $\mathbb{C}_+$, we see, by the Luzin-Privalov uniqueness theorem (see e.g. [36]), that

$$M_+(\eta)\ne M_-(\eta) \qquad \text{a.e. on } \mathbb{R}\ . \qquad (5.6)$$

Theorem 5.2. *Let the operator A be similar to a selfadjoint operator. Then, the following inequalities hold*

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t)-M_-(t)|^2}\left|g^+(t)\Sigma'_{ac+}(t)+(H(g^+\cdot d\Sigma_+)(t)\right|^2 dt \le K_1^+\int_{\mathbb{R}}|g^+(t)|^2 d\Sigma_+(t), \qquad (5.7)$$

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t)-M_-(t)|^2}\left|g^-(t)\Sigma'_{ac-}(t)+(H(g^-\cdot d\Sigma_-)(t)\right|^2 dt \le K_1^-\int_{\mathbb{R}}|g^-(t)|^2 d\Sigma_-(t), \qquad (5.8)$$

with constants K_1^+ and K_1^- independent of $g^\pm\in L^2(\mathbb{R}, d\Sigma_\pm)$.

Proof. Applying Fatou's theorem and using (2.33) we get

$$\lim_{\varepsilon\downarrow 0}\int \frac{g^\pm(t)}{t-(\eta+i\varepsilon)}d\Sigma_\pm(t) = \pi\cdot[g^\pm(\eta)\Sigma'_\pm(\eta)+iH(g^\pm d\Sigma_\pm)(\eta)] \qquad (5.9)$$

Passing to the limit in (5.1) (resp., (5.2)) as $\varepsilon\to 0$ and taking (5.9) into account we arrive at the inequality (5.7) (resp., (5.8)). ☐

Corollary 5.3. *Let the operator A be similar to a selfadjoint operator. Then*

$$\frac{\operatorname{Im} M_\pm(t)}{M_+(t)-M_-(t)} \in L^\infty(\mathbb{R})\ . \qquad (5.10)$$

Proof. Let A be similar to a selfadjoint operator. Then inequalities (5.7) and (5.8) hold. By Fatou Theorem $\pi\Sigma'_{ac\pm}(t) = \operatorname{Im} M_\pm(t+i0) =: \operatorname{Im} M_\pm(t)$ for a.e. $t \in \mathbb{R}$. Taking this relation into account and substituting in (5.7) (resp. (5.8)) any real-valued g^+_{ac} (resp. g^-_{ac}) with $g^\pm_{ac}(t) = 0$ for $t \in S'_s(\Sigma_\pm)$, we easily get

$$\int\limits_{\eta\in\mathbb{R}} \frac{(\operatorname{Im} M_\pm(\eta))^2}{|M_+(\eta) - M_-(\eta)|^2} |g^\pm_{ac}(\eta)|^2 \cdot \Sigma'_{ac\pm}(\eta) d\eta \le K_1^- \int_{\mathbb{R}} |g^\pm_{ac}(t)|^2 \cdot \Sigma'_{ac\pm}(t) dt.$$

Since this inequality holds for any $g^\pm_{ac} \in L^2(\mathbb{R}, d\Sigma_{ac\pm})$, we have

$$\frac{(\operatorname{Im} M_\pm(t))^2}{|M_+(t) - M_-(t)|^2} \in L^\infty(S'_{ac}(\Sigma_\pm)). \tag{5.11}$$

Inequality (5.11) yields (5.10) since $\operatorname{Im} M_\pm(t) = 0$ for a.e. $t \in \mathbb{R} \setminus S'_{ac}(\Sigma_\pm)$. ☐

Corollary 5.4. *Let the operator A be similar to a selfadjoint operator. Then, for all*

$$h^\pm \in L^2(\mathbb{R}) \cap L^2\left(\frac{1}{\Sigma'_{ac\pm}(t)}, \mathbb{R}\right), \quad h^\pm(t) = 0 \quad \textit{for} \quad t \in S'_s(\Sigma_\pm)\ ,$$

the following inequalities hold:

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2} \left|(Hh^+)(t)\right|^2 dt \le K_1^+ \int_{\mathbb{R}} |h^+(t)|^2 \frac{1}{\operatorname{Im} M_+(t)} dt, \tag{5.12}$$

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2} \left|(Hh^-)(t)\right|^2 dt \le K_1^+ \int_{\mathbb{R}} |h^-(t)|^2 \frac{1}{\operatorname{Im} M_-(t)} dt, \tag{5.13}$$

where K_1^+ and K_1^- are constants independent of $h^\pm$.

Proof. Inequality (5.7) yields

$$\int\limits_{\mathbb{R}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2} \left|(H(g^+ \cdot d\Sigma_+)(t)\right|^2 dt \le K_1^- \int_{\mathbb{R}} |g^+(t)|^2 d\Sigma_+(t), \tag{5.14}$$

Choosing any g^+_{ac} with $g^+_{ac}(t) = 0$ for $t \in S'_s(\Sigma_+)$, and setting in (5.14) $h^\pm := g^\pm \cdot (\Sigma'_{ac\pm})$ we arrive at the inequality (5.12). The inequality (5.13) is implied by (5.8) in just the same way. ☐

Corollary 5.5. *Let $E_\pm = \operatorname{supp} \Sigma'_{ac\pm}$ be the topological supports of measures $\Sigma_{ac\pm}$. If the operator A is similar to a selfadjoint operator, then*

$$\sup_{\mathcal{I}} \left(\frac{1}{|\mathcal{I} \cap E_\pm|} \int_{\mathcal{I}} \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2} dt\right) \cdot \left(\frac{1}{|\mathcal{I} \cap E_\pm|} \int_{\mathcal{I}} \operatorname{Im} M_\pm(t) dt\right) < \infty. \tag{5.15}$$

Proof. If A is similar to a selfadjoint operator, then by Corollary 5.4 two-weight estimates (5.12) and (5.13) for the Hilbert transform are valid. Due to (2.37) the result is immediate from (5.12) and (5.13). ☐

Due to Lebesgue theorem inequality (5.15) yields (5.10) and therefore gives another proof of Corollary 5.3. In fact, it gives a new necessary condition of similarity to a selfadjoint operator and is stronger than (5.10).

The following corollary gives one more necessary condition of similarity.

Corollary 5.6. *Let A be similar to a selfadjoint operator and let*

$$w_{1\pm}(t) := \frac{\operatorname{Im} M_\pm(t)}{|M_+(t) - M_-(t)|^2}.$$

Then

$$\sup_{\lambda\in\mathbb{C}_+} \mathcal{P}_\lambda(w_{1\pm}) \cdot \operatorname{Im} M_{ac\pm}(\lambda) = C < \infty, \tag{5.16}$$

where $M_{ac\pm}(\lambda) := \displaystyle\int_{\mathbb{R}} \frac{d\Sigma_{ac\pm}(t)}{t-\lambda}$, $\lambda \in \mathbb{C}_+$.

Proof. Note that $\operatorname{Im} M_\pm(t)$ is finite for a.e. $t \in \mathbb{R}$ and $\mathcal{P}_\lambda(\operatorname{Im} M_\pm) = \operatorname{Im} M_{ac\pm}(\lambda)$. We complete the proof by combining Corollary 5.4 with Proposition 2.8. □

Inequality (2.40) shows that condition (5.16) is stronger than (5.15).

Conjecture 5.1. *We conjecture that under the condition $\sigma_{disc}(A) = \emptyset$ inequalities* (5.7) *and* (5.8) *are also sufficient for the operator A to be similar to a selfadjoint operator. Therefore inequalities* (5.7) *and* (5.8) *reduce the similarity problem to two weight estimates for the Hilbert transform.*

Conjecture 5.2. *Suppose that $\sigma_{disc}(A) = \emptyset$ and both measures $d\Sigma_+$ and $d\Sigma_-$ are absolutely continuous, $\Sigma_\pm = \Sigma_{ac\pm}$. Then conditions* (5.12) *and* (5.13) *are sufficient for A to be similar to a selfadjoint operator.*

5.3 Sufficient conditions of similarity in terms of Weyl functions.

Consider an operator $\widetilde{A}$ given by $\widetilde{A} = A^*_{min} \upharpoonright \operatorname{dom}(\widetilde{A})$,

$$\operatorname{dom}(\widetilde{A}) = \{y \in \operatorname{dom}(A^*_{min}) : \ y(+0) = y(-0), \ y'(+0) = -y'(-0)\} . \tag{5.17}$$

Proposition 5.7. *The operator $\widetilde{A}$ is selfadjoint. For $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the resolvent of $\widetilde{A}$ has the form*

$$\mathcal{R}_{\widetilde{A}}(\lambda) f = \mathcal{R}_{A_0^-\oplus A_0^+}(\lambda) f + \widetilde{G}^-(\lambda)\psi_-(\lambda) + \widetilde{G}^+(\lambda)\psi_+(\lambda), \tag{5.18}$$

$$\widetilde{G}^+(\lambda) = -\widetilde{G}^-(\lambda) = \frac{1}{M_+(\lambda) + M_-(\lambda)} \int_{\mathbb{R}} \frac{g^-(t)d\Sigma_-(t) + g^+(t)d\Sigma_+(t)}{t-\lambda}, \tag{5.19}$$

where $g^\pm(t) = (\mathcal{F}_\pm f_\pm)(t)$, $f_\pm := P_\pm f \in L^2(\mathbb{R}_\pm)$.

Proof. Let $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^* := A^*_{min}$ defined by (2.24). Clearly, the extension $\widetilde{A}$ of A_{min} determined by (5.17), admits the following representation

$$\widetilde{A} = S^* | \operatorname{dom} \widetilde{A}, \qquad \operatorname{dom} \widetilde{A} = \ker(\Gamma_1 - B\Gamma_0), \qquad \text{where} \qquad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{5.20}$$

Thus, $\widetilde{A}$ is selfadjoint because so is B.

The representation (5.18) for the resolvent $\mathcal{R}_{\widetilde{A}}(\lambda)$ can be obtained in just the same way as representation for $\mathcal{R}_A(\lambda)$ in Proposition 4.4. □

Theorem 5.8. *Suppose that*

$$\sup_{\lambda\in\mathbb{C}_+}\frac{|M_+(\lambda)+M_-(\lambda)|}{|M_+(\lambda)-M_-(\lambda)|}\ <\ \infty. \tag{5.21}$$

Then the operator A is similar to a selfadjoint operator.

Proof. Since $\widetilde{A}$ and $A_0^-\oplus A_0^+$ are selfadjoint operators, we obtain from (5.18) and (5.3)

$$\varepsilon\int\limits_{\eta\in\mathbb{R}}\|\widetilde{G}^-(\eta+i\varepsilon)\ \psi_-(\eta+i\varepsilon)+\widetilde{G}^+(\eta+i\varepsilon)\ \psi_+(\eta+i\varepsilon)\|^2d\eta\le 4\pi\|f\|^2. \tag{5.22}$$

On the other hand, it follows from (5.19) with $f=f_\pm$ that

$$\|\widetilde{G}^\pm(\eta+i\varepsilon)\psi_\pm(\eta+i\varepsilon)\|^2= \tag{5.23}$$

$$=\frac{\operatorname{Im}M_+(\lambda)+\operatorname{Im}M_-(\lambda)}{\operatorname{Im}\lambda\cdot|M_+(\lambda)+M_-(\lambda)|^2}\left|\int_{\mathbb{R}}\frac{g^\pm(t)d\Sigma_\pm(t)}{t-\lambda}\right|^2. \tag{5.24}$$

Combining (5.22) with (5.23) we arrive at the following inequalities

$$\int\limits_{\eta\in\mathbb{R}}\frac{\operatorname{Im}M_\pm+\operatorname{Im}M_\mp(\lambda)}{|M_+(\lambda)+M_-(\lambda)|^2}\left|\int_{\mathbb{R}}\frac{g^\pm(t)d\Sigma_\pm(t)}{t-\lambda}\right|^2d\eta\le 4\pi\|f_\pm\|^2=4\pi\|g^\pm\|^2.$$

Combining these inequalities with (5.21) we arrive at estimates (5.1) and (5.2). Thus, by Theorem 5.1, A is similar to a selfadjoint operator. □

Remark 5.1. The condition (5.21) is not necessary for similarity to selfadjoint operator (see Remark 7.1)).

Remark 5.2. Note that sufficient condition (5.21) for similarity is weaker than either conditions (3.25)–(3.26) or conditions (3.28) obtained from Theorem 3.2 and Theorem 3.3, respectively. While these conditions guarantee a stronger result: similarity of A to an operator $B=B^*$ with absolutely continuous spectrum.

Finally, we apply Theorems 5.3 and 5.8 to the case of the operator A with constant potential. Consider a family of such operators

$$A(a):=(\operatorname{sgn}x)(-d^2/dx^2+a),\qquad a\in\mathbb{R}, \tag{5.25}$$

depending on a parameter a.

Proposition 5.9 ([28],[29]). *(i) The operator $A(a)$ is similar to a selfadjoint operator if and only if $a\ge 0$.*

(ii) The operator $A(0)$ is similar to the multiplication operator $Q:\ f\to xf(\cdot)$ in $L^2(\mathbb{R})$.

Proof. (i) In the case under consideration the functions $M_\pm(\lambda)$ are given by

$$M_\pm(\lambda)=\pm\frac{i}{\sqrt{\pm\lambda-a}}\ . \tag{5.26}$$

Since

$$M_+(\lambda) - M_-(\lambda) = \frac{i}{(\lambda - a)^{1/2}} + \frac{i}{(-\lambda - a)^{1/2}} \neq 0 \qquad \text{for} \quad \lambda \notin \mathbb{R},$$

Proposition 4.4 yields that the spectrum of $A(a)$ is real for any $a \in \mathbb{R}$ (see also [9]). It is clear that M_+ and M_- are holomorphic on $\mathbb{C} \setminus [a, +\infty)$ and $\mathbb{C} \setminus (-\infty, -a]$, respectively. Hence, by Proposition, we have 2.5 (iv) $\sigma(A(a)) = (-\infty, -a] \cup [a, +\infty)$, that is $\sigma(A(a)) = \mathbb{R}$ for $a \leq 0$ and $\sigma(A(a)) = \mathbb{R} \setminus (-a, a)$ for $a > 0$.

If $a \geq 0$, then the function

$$\frac{M_+(\lambda) + M_-(\lambda)}{M_+(\lambda) - M_-(\lambda)}$$

is bounded in $\mathbb{C}_+$. Thus, by Theorem 5.8, A is similar to a selfadjoint operator.

Now let $a < 0$. Setting $\lambda = i\varepsilon$ and $i\varepsilon - a = \rho e^{i\phi}$ we get

$$M_+(i\varepsilon) - M_-(i\varepsilon) = i\rho^{-1/2} \cdot [e^{-i\phi/2} - e^{i\phi/2}] = 2\rho^{-1/2} \sin(\phi/2),$$

and

$$\operatorname{Im} M_+(i\varepsilon) = \operatorname{Im}(i\rho^{-1/2} e^{i\phi/2}) = \rho^{-1/2} \cos(\phi/2).$$

Hence

$$\operatorname{Im} M_+(i\varepsilon)(M_+(i\varepsilon) - M_-(i\varepsilon))^{-1} = 2^{-1} \cot(\phi/2)$$

is unbounded in any neighborhood of zero. Thus, by Corollary 5.3 the operator A is not similar to a selfadjoint operator.

(ii) Let now $A = A(0)$. Substituting expressions (5.26) in formula (3.21) for $\theta_A(\cdot)$ and using the relation $\sqrt{\lambda}/\sqrt{-\lambda} = -i$, we arrive at the following formula for the characteristic function

$$\theta_A(\lambda) = \begin{pmatrix} -i & (i-1)/\sqrt{-\lambda} \\ (i-1)\sqrt{\lambda} & -i \end{pmatrix} . \tag{5.27}$$

It follows that $\theta_A(\cdot)$ is unbounded only near zero and infinity. Since the operator A has no eigenvalues, then by Proposition 3.4 (or by Corollary 3.5) it is similar to a selfadjoint operator $T_0 = T_0^*$ with absolutely continuous spectrum, $\sigma(T_0) = \sigma_{ac}(T_0) = \mathbb{R},\ \sigma_s(T_0) = \sigma_p(T_0) = \emptyset$. It is easily seen that the multiplicity of spectrum is one. Therefore T_0 is unitarily equivalent to the multiplication operator Q. $\square$

Remark 5.3. Using the Krein-Langer spectral theory of definitizable operators in Krein spaces Ćurgus and Langer [7] investigated the critical point ∞ of differential operators with an indefinite weight. Their results imply similarity of the operator $A(a)$ to a selfadjoint one if only $a > 0$.

The case $a = 0$ is more complicated since $A(0)$ has two critical points: zero and infinity. Similarity of $A(0)$ to a selfadjoint operator was established by Ćurgus and Najman [8] in the framework of Krein space approach.

Other proofs of the latter result have been obtained by several authors (see [27, 28, 15, 25]). In full generality statement (i) of Proposition 5.9 has originally been proved by one of the authors [29, 28], by using the resolvent criterion of similarity (see Theorem 3.12). The proof given above is similar to that contained in our short communication [33].

6 Restrictions of A to invariant subspaces corresponding to $\sigma_{disc}(A)$ and $\sigma_{ess}(A)$

Throughout this section we assume additionally that the following assumption is valid.

Assumption 6.1. *Suppose that the set* $\sigma_{disc}(A)$ *is finite.*

It will be shown in Section 7 that this condition is fulfilled if the potential q is finite-zone.

Since $\operatorname{dist}(\sigma_{ess}(A), \sigma_{disc}(A)) > 0$, we can apply the theorem on spectral decomposition (see [34, Theorem III.6.17]). That is there exists a skew decomposition $L^2(\mathbb{R}) = \mathfrak{H} = \mathfrak{H}_e \dot{+} \mathfrak{H}_d$ such that

$$A = A_{ess} \dot{+} A_{disc}, \quad A_{ess} = A \upharpoonright (\operatorname{dom}(A) \cap \mathfrak{H}_e), \quad A_{disc} = A \upharpoonright (\operatorname{dom}(A) \cap \mathfrak{H}_d) \tag{6.1}$$
$$\text{and} \quad \sigma(A_{disc}) = \sigma_{disc}(A), \quad \sigma(A_{ess}) = \sigma_{ess}(A).$$

We denote by P_e and P_d the corresponding skew projections onto $\mathfrak{H}_e$ and $\mathfrak{H}_d$, respectively.

Since $\sigma_{disc}(A)$ is finite, we see that A_{disc} is an operator in a finite dimensional space $\mathfrak{H}_d$. Jordan normal form of A_{disc} is described in Proposition 4.3 (3)-(5). By Proposition 4.3, we have $\sigma_{ess}(A) = \sigma_{ess}(A_2^-) \cup \sigma_{ess}(A_2^+)$. Thus $\sigma(A_{ess}) \subset \mathbb{R}$. This section is devoted to the question of similarity of A_{ess} to a selfadjoint operator.

Proposition 6.1. *Let Assumption 6.1 be fulfilled. Suppose G_d be a compact subset of $\mathbb{C}$ such that $G_d \cap \sigma_{ess}(A) = \emptyset$ and all $\lambda \in \sigma_{disc}(A)$ are interior points of G_d. Suppose $Q^\pm$ are dense subsets in $L^2(\mathbb{R}, d\Sigma_\pm)$.*

Then the following conditions are equivalent:

(i) *the operator A_{ess} in $\mathfrak{H}_e$ is similar to a selfadjoint one;*

(ii) *the operator $A_{ess}P_e$ in $\mathfrak{H} = L^2(\mathbb{R})$ is similar to a selfadjoint one;*

(iii) *the inequality*

$$\varepsilon \cdot \int_{\mathbb{R}} \|\mathcal{R}_A(\eta + i\varepsilon) f_e\|^2 d\eta \le C_1^e \|f_e\|^2 \tag{6.2}$$

holds for all $\varepsilon > 0$, $f_e \in \mathfrak{H}_e$ with some constant C_1^e.

(iv) *for all $\varepsilon > 0$ and $g^\pm \in Q^\pm$ the following inequalities hold:*

$$\int\limits_{\substack{\eta \in \mathbb{R} \\ \eta + i\varepsilon \notin G_d}} \frac{\operatorname{Im} M_\pm(\eta + i\varepsilon)}{|M_+(\eta + i\varepsilon) - M_-(\eta + i\varepsilon)|^2} \left| \int_{\mathbb{R}} \frac{g^-(t) d\Sigma_-(t)}{t - (\eta + i\varepsilon)} \right|^2 d\eta \le C_2^- \|g^-\|^2_{L^2(d\Sigma_-)}, \tag{6.3}$$

$$\int\limits_{\substack{\eta \in \mathbb{R} \\ \eta + i\varepsilon \notin G_d}} \frac{\operatorname{Im} M_\pm(\eta + i\varepsilon)}{|M_+(\eta + i\varepsilon) - M_-(\eta + i\varepsilon)|^2} \left| \int_{\mathbb{R}} \frac{g^+(t) d\Sigma_+(t)}{t - (\eta + i\varepsilon)} \right|^2 d\eta \le C_2^+ \|g^+\|^2_{L^2(d\Sigma_+)}, \tag{6.4}$$

where $C_2^\pm$ are constants independent of ε and $g^\pm$.

Proof. It is clear that $(i) \Leftrightarrow (ii)$.

Let us show that $(ii) \Leftrightarrow (iii)$. It can easily be checked that $A_{ess}P_e$ is a J-selfadjoint operator (see [37]). By Proposition 3.13, assertion (ii) holds if and only if for all $\varepsilon > 0$ and $f \in \mathfrak{H}$ the following inequality holds

$$\varepsilon \int_{\mathbb{R}} \|\mathcal{R}_{AP_e}(\eta + i\varepsilon) f\|^2 d\eta \leq C_1 \|f\|^2 \ , \quad C_1 = const. \tag{6.5}$$

Clearly, (6.5) is equivalent to (6.2).

Now we show that $(iii) \Leftrightarrow (iv)$. Let $f \in L^2(\mathbb{R})$, $f_e = P_e f$, $f_d = P_d f$. It can be shown in the usual way that there exist constants C_2, C_3 such that $\|\mathcal{R}_A f_e\| = \|\mathcal{R}_{A_{ess}} f_e\| \leq C_2 \|f_e\|$ for $\lambda \in G_d$, and $\|\mathcal{R}_A f_d\| = \|\mathcal{R}_{A_{disc}} f_d\| \leq \dfrac{C_3}{1+|\lambda|}\|f_d\|$ for $\lambda \in \mathbb{C} \setminus G_d$. Therefore (6.5) is equivalent to

$$\varepsilon \int_{\substack{\eta \in \mathbb{R} \\ \eta + i\varepsilon \notin G_d}} \|\mathcal{R}_A(\eta + i\varepsilon) f\|^2 d\eta \leq C_1 \|f\|^2, \quad \forall f \in L^2(\mathbb{R}), \forall \varepsilon > 0. \tag{6.6}$$

Arguing as in the proof of the Theorem 5.1, we see that condition (6.6) is fulfilled iff the inequalities (6.3) and (6.4) hold for all $g^{\pm} \in L^2(d\Sigma_{\pm})$ and $\varepsilon > 0$.

We show that it suffices to check (6.3) and (6.4) only for dense subsets $Q^{\pm}$.

Let $\varepsilon > 0$ be a fixed positive number, $\mathcal{I}$ an open bounded set in $\mathbb{R}$. Denote $\mathcal{I}_\varepsilon := \{\eta + i\varepsilon : \eta \in \mathcal{I}\}$. Assume that $\mathcal{I}_\varepsilon \cap G_d = \emptyset$. Then $(M_+(\lambda) - M_-(\lambda))^{-1}$ is holomorphic on $\mathcal{I}_\varepsilon$. By the Schwarz inequality, the operators

$$K^{\pm}_{\mathcal{I}_\varepsilon} : g^+ \mapsto \frac{(\operatorname{Im} M_{\pm}(\eta + i\varepsilon))^{1/2}}{M_+(\eta + i\varepsilon) - M_-(\eta + i\varepsilon)} \int_{\mathbb{R}} \frac{g^+(t) d\Sigma_+(t)}{t - (\eta + i\varepsilon)},$$

are bounded from $L^2(\mathbb{R}, d\Sigma_+)$ to $L^2(\mathcal{I}_\varepsilon, d\eta)$.

Suppose that Q^+ is dense in $L^2(d\Sigma_+)$ and (6.4) is fulfilled for Q^+. Then $\|K^{\pm}_{\mathcal{I}_\varepsilon}\| \leq C_2^+$ for all $\varepsilon > 0$ and for all $\mathcal{I}$. This imply (6.4) for all $g^{\pm} \in L^2(d\Sigma_{\pm})$ and $\varepsilon > 0$. In the same way we can prove that (6.3) is equivalent to the inequality (6.3) for all $g^{\pm} \in L^2(d\Sigma_{\pm})$. $\square$

Recall that $\sigma_{ac}(T)$ and $\sigma_s(T)$ are the absolutely continuous and singular spectra of a selfadjoint operator T. Evidently,

$$\sigma_{ac}(A_2^{\pm}) = \operatorname{supp} d\Sigma_{ac\pm}, \qquad \sigma_s(A_2^{\pm}) = \operatorname{supp}(d\Sigma_{sc\pm} + d\Sigma_{d\pm}).$$

Note that $\sigma_{ac}(A_0^{\pm}) \subset \sigma_{ess}(A_0^{\pm})$. Therefore, by Proposition 4.3, we have

$$\operatorname{supp} d\Sigma_{ac-} \cup \operatorname{supp} d\Sigma_{ac+} = \sigma_{ac}(A_0^- \oplus A_0^+) \subset \sigma_{ess}(A). \tag{6.7}$$

Proposition 6.2. *Let Assumption 6.1 be fulfilled. Suppose the operator A_{ess} is similar to a selfadjoint operator. Then*

$$\frac{\operatorname{Im} M_{ac\pm}(t)}{M_+(t) - M_-(t)} \in L^\infty(\mathbb{R}). \tag{6.8}$$

Taking into account (6.7), we see that this theorem can be proved in the same way as Theorem 5.3.

Assumption 6.2. *In what follows we assume that*

$$d\Sigma_- = d\Sigma_{ac-} + d\Sigma_{d-}, \quad \operatorname{supp} d\Sigma_{d-} = \{\theta_j^-\}_{j=1}^{N_\theta^-}, \quad N_\theta^- < \infty,$$
$$\text{and} \qquad d\Sigma_+ = d\Sigma_{ac+} + d\Sigma_{d+}, \quad \operatorname{supp} d\Sigma_{d+} = \{\theta_j^+\}_{j=1}^{N_\theta^+}, \quad N_\theta^+ < \infty.$$

Then $M_\pm(\lambda) = M_{ac\pm}(\lambda) + M_{d\pm}(\lambda)$, where

$$M_{ac\pm}(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma_{ac\pm}(t)}{t-\lambda}\ , \qquad \text{and} \qquad M_{d\pm}(\lambda) = \sum_{j=1}^{N_\theta^\pm} \frac{\Sigma_\pm(\theta_j^\pm + 0) - \Sigma_\pm(\theta_j^\pm - 0)}{\theta_j^\pm - \lambda}\ .$$

.

Let us introduce the sets

$$\{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm} := \{\theta_j^\pm\}_1^{N_\theta^\pm} \setminus \sigma_{disc}(A); \tag{6.9}$$

here $\widetilde{N}_\theta^\pm < \infty$. (these sets will be used in Theorem 6.3).

Recall that we denote the Smirnov class on $\mathbb{C}_+$ (see Subsection 2.6) by $\mathcal{N}^+(\mathbb{C}_+)$.

Theorem 6.3. *Let Assumptions 6.1 and 6.2 be fulfilled. Let G_d be the compact set from Proposition 6.1.*

Suppose there exist functions $U_+(\lambda)$ and $U_-(\lambda)$ on $\mathbb{C}_+$ such that the following conditions hold:

$$\frac{\operatorname{Im} M_{ac\pm}(\lambda)}{|M_+(\lambda) - M_-(\lambda)|^2} \le C_\pm^u |U_\pm(\lambda)|^2, \quad \lambda \in \mathbb{C}_+ \setminus G_d\ , \tag{6.10}$$

$$U_\pm(\lambda) \in \mathcal{N}^+(C_+), \tag{6.11}$$

$$\frac{U_\pm(t)}{\theta_j^- - t} \in L^2(\mathbb{R}),\ j = 1, \cdots, N_\theta^-; \quad \frac{U_\pm(t)}{\theta_j^+ - t} \in L^2(\mathbb{R}),\ j = 1, \cdots, N_\theta^+\ , \tag{6.12}$$

where $C_\pm^u$ are constants.

Suppose there exist functions $w_+(\cdot)$ and $w_-(\cdot)$ on $\mathbb{R}$, $w_\pm(t) > 0$ a.e., such that the following conditions hold:

$$w_\pm(t) \le C_\pm^w (\Sigma'_{ac\pm}(t))^{-1} \quad \text{a.e. on} \quad \operatorname{supp} d\Sigma_{ac\pm}, \tag{6.13}$$

$$w_+(t) \ \text{ and } \ w_-(t) \quad \text{satisfy the } (A_2) \text{ condition (see (2.35))}, \tag{6.14}$$

$$\frac{U_+^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R}), \qquad \frac{U_-^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R})\ ; \tag{6.15}$$

where $C_\pm^w$ are constants.

Suppose that for every point $\widetilde{\theta}_j^\pm$ of the set $\{\widetilde{\theta}_k^\pm\}_1^{\widetilde{N}_\theta^\pm}$, there exist a function $U_j^\pm(\lambda) \in \mathcal{N}^+(C_+)$ and a neighborhood $D_j^\pm$ of the point $\widetilde{\theta}_j^\pm$ such that the following conditions hold:

$$\frac{1}{|M_+(\lambda) - M_-(\lambda)|^2} \operatorname{Im} \frac{1}{\widetilde{\theta}_j^\pm - \lambda} \le C_\theta^u |U_j^\pm(\lambda)|^2 \quad \text{for } \lambda \in D_j^\pm \cap \mathbb{C}_+\ , \tag{6.16}$$

$$\frac{|U_j^{\pm}(t)|^2}{w_+(t)} \in L^\infty(\mathbb{R}), \quad \frac{|U_j^{\pm}(t)|^2}{w_-(t)} \in L^\infty(\mathbb{R}) \ , \tag{6.17}$$

$$\frac{1}{|\widetilde{\theta}_j^{\pm} - \lambda|\,|M_+(\lambda) - M_-(\lambda)|} \le C_\theta^M \quad \textit{for } \lambda \in D_j^{\pm} \cap \mathbb{C}_+, \tag{6.18}$$

where C_θ^u *and* C_θ^M *are constants.*

Then A_{ess} *is similar to a selfadjoint operator.*

Proof. Let us show that (6.3) and (6.4) hold.

Let $\lambda = \eta + i\varepsilon$, $\eta = \operatorname{Re}\lambda$, $\varepsilon = \operatorname{Im}\lambda$.

1) Denote

$$\mathcal{I}_{\pm}(\varepsilon) := \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin G_d}} \frac{\operatorname{Im} M_{ac\pm}(\lambda)}{|M_+(\lambda) - M_-(\lambda)|^2} \left| \int_{\mathbb{R}} \frac{g^+(s) d\Sigma_+(s)}{s-\lambda} \right|^2 d\eta, \quad g^+ \in L^2(d\Sigma_+(t)) \ .$$

Let

$$Q_{ac}^+ := \{g^+ \in L^2(\mathbb{R}, d\Sigma_{ac+}(t)) : \ (g^+ \Sigma'_{ac+}) \in L^2(\mathbb{R}, dt)\}.$$

Then the set $Q^+ := Q_{ac}^+ \oplus L^2(\mathbb{R}, d\Sigma_{d+})$ is dense in $L^2(\mathbb{R}, d\Sigma_+(t))$.

First we show that

$$\mathcal{I}_{\pm}(\varepsilon) \le C_2^+ \|g^+\|^2_{L^2(d\Sigma_+)} \quad \text{for } g^+ \in Q^+. \tag{6.19}$$

Let us denote

$$K_{\pm}^{ac}(\lambda) := U_{\pm}(\lambda) \int_{\mathbb{R}} \frac{g^+(t) d\Sigma_{ac+}(t)}{t-\lambda}, \qquad K_{\pm}^{d}(\lambda) := U_{\pm}(\lambda) \int_{\mathbb{R}} \frac{g^+(t) d\Sigma_{d+}(t)}{t-\lambda},$$
$$K_{\pm}(\lambda) := K_{\pm}^{ac}(\lambda) + K_{\pm}^{d}(\lambda) = U_{\pm}(\lambda) \int_{\mathbb{R}} \frac{g^+(t) d\Sigma_+(t)}{t-\lambda}.$$

By (6.10), we have

$$\mathcal{I}_{\pm}(\varepsilon) \le \int_{\mathbb{R}} |K_{\pm}(\lambda)|^2 d\eta. \tag{6.20}$$

It follows from $U_{\pm}(\lambda) \in \mathcal{N}^+(\mathbb{C}_+)$ that $U_{\pm}(\lambda)$ is holomorphic in $\mathbb{C}_+$ and has the nontangential limit $U_{\pm}(\eta)$ for almost all $\eta \in \mathbb{R}$ (see [18]). Since $g^+ \in Q^+$, we have $g^+(t)\Sigma'_{ac+}(t) \in L^2(\mathbb{R}, dt)$. Therefore,

$$\int_{\mathbb{R}} \frac{g^+(t) d\Sigma_{ac+}(t)}{t-\lambda} \in H^2(\mathbb{C}_+) \ .$$

It follows from [18, Corollary II.5.6] and [18, Corollary II.5.7] that $K_{\pm}^{ac}(\lambda) \in \mathcal{N}^+(\mathbb{C}_+)$. The functions $(\theta_j^+ - \lambda)^{-1}$ are outer in $\mathbb{C}_+$. Therefore [18, Corollary II.5.6] and Lemma 2.10 yield $K_{\pm}^d(\lambda) \in \mathcal{N}^+(\mathbb{C}_+)$. Hence $K_{\pm}^{ac}(\lambda)$, $K_{\pm}^d(\lambda)$, and $K_{\pm}(\lambda)$ belong to $\mathcal{N}^+(\mathbb{C}_+)$ and have the nontangential limits $K_{\pm}^{ac}(\eta)$, $K_{\pm}^d(\eta)$ and $K_{\pm}(\eta)$ for almost all $\eta \in \mathbb{R}$. Note also that

$$K_{\pm}^{ac}(\eta) := \pi U_{\pm}(\eta) \left(g^+(\eta)\Sigma'_{ac+}(\eta) + \mathcal{H}(g^+\Sigma'_{ac+})(\eta) \right) \quad \text{for a.e. } \eta \in \mathbb{R}. \tag{6.21}$$

Assume that the following inequality holds

$$\int_{\mathbb{R}} \|K_\pm(\eta)\|^2 d\eta \le C_2^+ \|g^+\|^2_{L^2(d\Sigma_+)}. \tag{6.22}$$

Then, by [18, Section II.5], we have $K_\pm(\lambda) \in H^2(\mathbb{C}_+)$ and for all $\varepsilon > 0$

$$\int_{\mathbb{R}} \|K_\pm(\eta + i\varepsilon)\|^2 d\eta \le \|K_\pm(\lambda)\|^2_{H^2(\mathbb{C}_+)} = \|K_\pm(\eta)\|^2_{L^2(\mathbb{R})} \le C_2^+ \|g^+\|^2_{L^2(d\Sigma_+)} .$$

Combining this with (6.20), we see that (6.22) yields (6.19) with a constant C_2^+ independent of $g^+ \in Q^+$.

Let us prove (6.22). By (6.12), we have

$$\begin{aligned}\|K^d_\pm(\eta)\|_{L^2(\mathbb{R})} \le C_3^\pm \sum_{j=1}^{N_\theta^+} g^+(\theta_j^+) \left(\Sigma_+(\theta_j^+ + 0) - \Sigma_+(\theta_j^+ - 0)\right)^{1/2} \le \\ \le \text{where } C_3^\pm \sqrt{N_\theta^+}\ \|g^+\|_{L^2(d\Sigma_+)},\end{aligned} \tag{6.23}$$

$$C_3^\pm = \max\left\{ (\Sigma_+(\theta_j^+ + 0) - \Sigma_+(\theta_j^+ - 0))^{1/2} \left\| \frac{U_\pm(\eta)}{\theta_j^+ - \eta} \right\|_{L^2(\mathbb{R})} \right\}_{j=1}^{N_\theta^+} < \infty.$$

It follows from (6.13) that

$$\|g^+(t)\Sigma'_{ac+}(t)\|^2_{L^2(w_+(t)dt)} \le C_+^w \|g^+\|^2_{L^2(d\Sigma_{ac+})} \le C_+^w \|g^+\|^2_{L^2(d\Sigma_+)} . \tag{6.24}$$

Since $w_+(t) \in (A_2)$, we have

$$\|\mathcal{H}(g^+\Sigma'_{ac+})(t)\|^2_{L^2(w_+(t)dt)} \le C_1 \|g^+(t)\Sigma'_{ac+}(t)\|^2_{L^2(w_+(t)dt)}, \tag{6.25}$$

where C_1 is a constant independent of g^+. It follows from (6.25) and (6.21) that

$$\begin{aligned}\int_{\mathbb{R}} |K^{ac}_\pm(\eta)|^2 d\eta \le \\ \le \left\| \frac{U_\pm^2(\eta)}{w_+(\eta)} \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |g^+(\eta)\Sigma'_{ac+}(\eta) + \mathcal{H}(g^+\Sigma'_{ac+})(\eta)|^2 w_+(\eta) d\eta \le \\ \le 2(1 + C_1) \left\| \frac{U_\pm^2(\eta)}{w_+(\eta)} \right\|_{L^\infty(\mathbb{R})} \|g^+(\eta)\Sigma'_{ac+}(\eta)\|^2_{L^2(w_+(\eta)d\eta)} .\end{aligned} \tag{6.26}$$

Combining (6.26), (6.15), and (6.24), we get

$$\int_{\mathbb{R}} |K^{ac}_\pm(\eta)|^2 d\eta \le C_2 \|g^+\|^2_{L^2(d\Sigma_+)} ,$$

where the constant C_2 is independent of g^+. Taking into account (6.23), we obtain (6.22).

Let us remember that (6.22) implies (6.19). Thus (6.19) is proved.

2) Denote

$$\mathcal{I}_{d\pm}(\varepsilon) := \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin G_d}} \frac{\operatorname{Im} M_{d\pm}(\lambda)}{|M_+(\lambda)-M_-(\lambda)|^2} \left| \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_+(s)}{s-\lambda} \right|^2 d\eta.$$

Let us show that

$$\mathcal{I}_{d\pm}(\varepsilon) \le C_3 \|g^+\|^2_{L^2(d\Sigma_+)} \quad \text{for } g^+ \in Q^+ \tag{6.27}$$

(here and below C_3, C_4, ... are some constants). It is suffices to prove the inequality (6.27) for each summand, i.e.,

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin G_d}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2} \operatorname{Im}\frac{1}{\theta_j^\pm-\lambda} \left| \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_+(s)}{s-\lambda} \right|^2 d\eta \le C_4 \|g^+\|^2_{L^2(d\Sigma_+)} \tag{6.28}$$

for $j = 1, \ldots, N_\theta^\pm$.

Assume $\theta_j^\pm \in \sigma_{disc}(A)$. Then

$$\operatorname{Im}\frac{1}{\theta_j^\pm-\lambda} \le C_5 \operatorname{Im} M_{ac\pm}(\lambda), \quad \lambda \in \mathbb{C}_+ \setminus G_d\ .$$

Thus (6.28) follows from (6.19).

Assume $\theta_j^\pm \notin \sigma_{disc}(A)$. In this case, $\theta_j^\pm \in \{\widetilde{\theta}_k^\pm\}_1^{\widetilde{N}_\theta}$. Let k be such that $\theta_j^\pm = \widetilde{\theta}_k^\pm$. By assumptions of the theorem, conditions (6.16), (6.17), and (6.18) hold. It is easy to see that

$$\operatorname{Im}\frac{1}{\widetilde{\theta}_k^\pm-\lambda} \le C_6 \operatorname{Im} M_{ac\pm}(\lambda), \quad \lambda \in \mathbb{C}_+ \setminus D_k^\pm\ .$$

Therefore (6.19) implies

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\notin (D_k^\pm\cup G_d)}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2} \operatorname{Im}\frac{1}{\widetilde{\theta}_k^\pm-\lambda} \left| \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_+(s)}{s-\lambda} \right|^2 d\eta \le$$

$$\le C_4 \|g^+\|^2_{L^2(d\Sigma_+)}\ . \tag{6.29}$$

By (6.18), we have

$$\int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^\pm}} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2} \operatorname{Im}\frac{1}{\widetilde{\theta}_k^\pm-\lambda} \left| \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda} \right|^2 d\eta \le$$

$$\le \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^\pm}} \frac{\varepsilon}{(\widetilde{\theta}_k^\pm-\eta)^2+\varepsilon^2} \frac{1}{|M_+(\lambda)-M_-(\lambda)|^2|\widetilde{\theta}_k^\pm-\lambda|^2} \left| (\widetilde{\theta}_k^\pm-\lambda) \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda} \right|^2 d\eta \le$$

$$\le C_\theta^M \int\limits_{\substack{\eta\in\mathbb{R}\\ \eta+i\varepsilon\in D_k^\pm}} \frac{\varepsilon}{(\widetilde{\theta}_k^\pm-\eta)^2+\varepsilon^2} \left| (\widetilde{\theta}_k^\pm-\lambda) \int_{\mathbb{R}} \frac{g^+(s)d\Sigma_{d+}(s)}{s-\lambda} \right|^2 d\eta. \tag{6.30}$$

We may assume that
$$D_k^{\pm} \cap \left(\{\theta_j^+\}_1^{N_\theta^+} \cup \{\theta_j^-\}_1^{N_\theta^-} \right) = \{\widetilde{\theta}_k^{\pm}\} \ .$$
Therefore,
$$\left| (\widetilde{\theta}_k^{\pm} - \lambda) \int_{\mathbb{R}} \frac{g^+(s) d\Sigma_{d+}(s)}{s-\lambda} \right| \le C_7 \|g^+\|^2_{L^2(d\Sigma_+)}, \quad \lambda \in D_k^{\pm} \ .$$
If we combine this with properties of Poisson kernel (see [18, Section I.3]), we get
$$\int\limits_{\substack{\eta\in\mathbb{R} \\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{\varepsilon}{(\widetilde{\theta}_k^{\pm} - \eta)^2 + \varepsilon^2} \left| (\widetilde{\theta}_k^{\pm} - \lambda) \int_{\mathbb{R}} \frac{g^+(s) d\Sigma_{d+}(s)}{s-\lambda} \right|^2 d\eta \le \pi C_7 \|g^+\|^2_{L^2(d\Sigma_+)}. \tag{6.31}$$
Using (6.31) and (6.30), we get
$$\int\limits_{\substack{\eta\in\mathbb{R} \\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{1}{|M_+(\lambda) - M_-(\lambda)|^2} \operatorname{Im} \frac{1}{\widetilde{\theta}_k^{\pm} - \lambda} \left| \int_{\mathbb{R}} \frac{g^+(s) d\Sigma_{d+}(s)}{s-\lambda} \right|^2 d\eta \le \pi C_\theta^M C_7 \|g^+\|^2_{L^2(d\Sigma_+)} \ . \tag{6.32}$$
The inequality
$$\int\limits_{\substack{\eta\in\mathbb{R} \\ \eta+i\varepsilon\in D_k^{\pm}}} \frac{1}{|M_+(\lambda) - M_-(\lambda)|^2} \operatorname{Im} \frac{1}{\widetilde{\theta}_k^{\pm} - \lambda} \left| \int_{\mathbb{R}} \frac{g^+(s) d\Sigma_{ac+}(s)}{s-\lambda} \right|^2 d\eta \le C_9 \|g^+\|^2_{L^2(d\Sigma_+)} \tag{6.33}$$
follows from (6.16), (6.17), and (6.14) in the same way as (6.26) follows from (6.10), (6.12), and (6.14).

Combining (6.33), (6.32) and (6.29), we get (6.28). Thus (6.27) is proved. Inequality (6.4) is proved. Inequality (6.3) can be shown in the same way. Thus Proposition 6.1 yields that A_{ess} is similar to a selfadjoint operator. □

7 Indefinite Sturm-Liouville operators with finite-zone potentials

7.1 Spectral properties of A_{ess} and A_{disc}

Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with a finite-zone potential q (see Subsection 2.4).

In this case, we have
$$\sigma(A_0^{\pm}) = \sigma_{ac}(A_0^{\pm}) \cup \sigma_{disc}(A_0^{\pm}), \tag{7.1}$$

$$\sigma_{ac}(A_0^+) = -\sigma_{ac}(A_0^-) = \sigma(L) = [\overset{r}{\mu}_0, \overset{l}{\mu}_1] \cup [\overset{r}{\mu}_1, \overset{l}{\mu}_2] \cup \dots \cup [\overset{r}{\mu}_N, +\infty), \tag{7.2}$$

$$\sigma_{disc}(A_0^{\pm}) = \{\pm\tau_j : \tau_j \notin \{\overset{r}{\mu}_k\}_0^N \cup \{\overset{l}{\mu}_k\}_1^N, \ Q(\tau_j) \pm i\sqrt{R(\tau_j)} \ne 0\} =: \{\theta_k^{\pm}\}_1^{N_\theta^{\pm}}. \tag{7.3}$$

Let $M(z)$ be a (multivalued) analytical function. If $M(z) = \sum\limits_{k=-\infty}^{+\infty} m_k(z-a)^{k/n}$ in a sufficiently small neighborhood of a point $a \in \mathbb{C}$, then we say that the number

$$\frac{1}{n}\inf\{k : m_k \neq 0\} \quad (-\frac{1}{n}\inf\{k : m_k \neq 0\})$$

is *the generalized order of a zero (pole) of the function* $M(z)$ *at the point* a. Recall that the functions $M_\pm(\lambda)$ are holomorphic in $\rho(A_0^\pm)$. For $\eta \in \sigma(A_0^\pm)$, we set $M_\pm(\eta) := M_\pm(\eta + i0)$. Note that in the case of a finite-zone potential q the functions $M_\pm(\lambda)$ can be continued on $\mathbb{C}$ as multivalued analytical functions with finite number of poles and finite number of branch points. Let us denote these continuations by $\widehat{M}_\pm(\lambda)$. Then

$\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N$ are the sets of branch points for $\widehat{M}_\pm(\lambda)$;

$\{\pm\xi_j\}_1^N \cap (\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N)$ are the sets of zeroes of the generalized order $1/2$ for $\widehat{M}_\pm(\lambda)$;

$\{\pm\tau_j\}_0^N \cap (\{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N)$ are the sets of poles of the generalized order $1/2$ for $\widehat{M}_\pm(\lambda)$;

$\{\theta_j^\pm\}_1^{N_\theta^\pm}$ are the sets of poles of the first order for $\widehat{M}_\pm(\lambda)$;

$\{\pm\xi_j : \xi_j \notin \{\overset{r}{\mu}_k\}_0^N \cup \{\overset{l}{\mu}_k\}_1^N,\ Q(\xi_j) \mp i\sqrt{R(\xi_j)} \neq 0\}$ are the sets of zeroes of the first order for $\widehat{M}_\pm(\lambda)$.

We will say that λ_0 is *a generalized zero (pole)* of $M_\pm$ if the generalized order of a zero (a pole) at λ_0 is positive.

We denote by $\overset{*}{M}_\pm$ the holomorphic continuation of $M_\pm(\lambda)$ from $\mathbb{C}_+$ to

$$\mathbb{C} \setminus \left\{\lambda :\ \operatorname{Im}\lambda < 0,\ \operatorname{Re}\lambda \in \{\pm\overset{r}{\mu}_j\}_0^N \cup \{\pm\overset{l}{\mu}_j\}_1^N\right\}.$$

Theorem 7.1. *Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with a finite-zone potential q. Let $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$. Then:*

1) *The operator A has finite number of eigenvalues,*

$$\sigma_p(A) = \left(\{\theta_j^+\}_1^{N_\theta^+} \cap \{\theta_j^-\}_1^{N_\theta^-}\right) \cup \{\lambda \in \rho(A_2^+ \oplus A_2^-) : M_+(\lambda) = M_-(\lambda)\}. \tag{7.4}$$

2) *The eigenvalues of A are isolated and have finite algebraic multiplicity, the geometric multiplicity equals one for all eigenvalues of A.*

3) *If $\lambda_0 \in \rho(A_2^+) \cap \rho(A_2^-)$, then the algebraic multiplicity of λ_0 is equal to the multiplicity of λ_0 as a zero of the holomorphic function $M_+(\lambda) - M_-(\lambda)$; if $\lambda_0 \in \{\theta_j^+\}_1^{N_\theta^+} \cap \{\theta_j^-\}_1^{N_\theta^-}$, then the algebraic multiplicity of λ_0 is equal to the multiplicity of λ_0 as a zero of the holomorphic function $\dfrac{1}{M_+(\lambda)} - \dfrac{1}{M_-(\lambda)}$.*

4) *There exist a skew decomposition $L^2(\mathbb{R}) = \mathfrak{H}_e \dot{+} \mathfrak{H}_d$ such that*

$$A = A_{ess} \dot{+} A_{disc}, \quad A_{ess} = A \upharpoonright (\mathrm{dom}(A) \cap \mathfrak{H}_e), \quad A_{disc} = A \upharpoonright (\mathrm{dom}(A) \cap \mathfrak{H}_d),$$
$$\sigma(A_{disc}) = \sigma_{disc}(A), \quad \sigma(A_{ess}) = \sigma_{ess}(A). \tag{7.5}$$

Besides, $\mathfrak{H}_d$ is a finite-dimensional space.

Proof. The spectral functions $\Sigma_\pm$ have the forms $\Sigma_\pm(t) = \Sigma_{ac\pm}(t) + \Sigma_{d\pm}(t)$, where

$$\Sigma'_{ac\pm}(t) = \begin{cases} \frac{\sqrt{R(\pm t)}}{S(\pm t)}, & t \in \pm \bigcup_{j=0}^{N} (\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1}) \\ 0, & t \notin \pm \bigcup_{j=0}^{N} [\overset{l}{\mu}_j, \overset{r}{\mu}_j] \end{cases}. \tag{7.6}$$

Here the branch of multifunction $\sqrt{R(\pm\lambda)}$ is chosen such that $\Sigma'_{ac\pm}(t) \geq 0$ a.e. (see [24], [38]).
Consequently,

$$\Sigma'_{ac\pm}(t) \asymp 1 \quad (t \to t_0), \quad t_0 \in \pm \bigcup_{j=0}^{N} (\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1}), \tag{7.7}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{1/2} \chi_\pm(t - t_0) \quad (t \to t_0), \quad t_0 \in \{\pm \overset{r}{\mu}_j\}_0^N \setminus \{\pm \tau_j\}_0^N, \tag{7.8}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{1/2} \chi_\mp(t - t_0) \quad (t \to t_0), \ t_0 \in \{\pm \overset{l}{\mu}_j\}_1^N \setminus \{\pm \tau_j\}_0^N, \tag{7.9}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{-1/2} \chi_\pm(t - t_0) \quad (t \to t_0), \ t_0 \in \{\pm \overset{r}{\mu}_j\}_0^N \cap \{\pm \tau_j\}_0^N, \tag{7.10}$$

$$\Sigma'_{ac\pm}(t) \asymp |t - t_0|^{-1/2} \chi_\mp(t - t_0) \quad (t \to t_0), \ t_0 \in \{\pm \overset{l}{\mu}_j\}_1^N \cap \{\pm \tau_j\}_0^N. \tag{7.11}$$

Therefore,

$$\int\limits_{\mathbb{R}\setminus\{\eta_0\}} \frac{1}{|t - \eta_0|^2} d\Sigma_\pm(t) = \infty, \quad \forall \eta_0 \in \pm(\cup_{j=0}^{N} [\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1}] \cup [\overset{r}{\mu}_N, +\infty)) .$$

Combining this with Theorem 4.2 (1), we get

$$\sigma_p(A) \subset \mathbb{C} \setminus (\sigma_{ac}(A_2^+) \cup \sigma_{ac}(A_2^-)) = \mathbb{C} \setminus \sigma_{ess}(A).$$

Thus, Proposition 4.3 yields (7.4).

Taking into account (2.17), we can write the equation $M_+(\lambda) = M_-(\lambda)$ in the form

$$\frac{P(\lambda)}{Q(\lambda) - i\sqrt{R(\lambda)}} = \frac{P(-\lambda)}{Q(-\lambda) + i\sqrt{R(-\lambda)}},$$

where P, Q, and R are polynomials. Thus the equation $M_+(\lambda) = M_-(\lambda)$ has finite number of solutions. Therefore the set $\sigma_p(A)$ is finite. Statement (1) is proved.

Statements (2) and (3) follow from Statement (1) and Proposition 4.3. Statement (4) follows from statements(1), (2), and (6.1). $\square$

Theorem 7.2. *Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with a finite-zone potential, let $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$. Then the following statements are equivalent:*

(i) *The operator A_{ess} is similar to a selfadjoint operator;*

(ii) *The following conditions are satisfied*

$$\frac{\operatorname{Im} M_\pm}{M_+(t) - M_-(t)} \in L^\infty(\mathbb{R}); \tag{7.12}$$

(iii) *The function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ has no generalized zeroes in*

$$(-\infty, -\overset{r}{\mu}_N) \cup (-\overset{l}{\mu}_N, -\overset{r}{\mu}_{N-1}) \cup \cdots \cup (-\overset{l}{\mu}_1, -\overset{r}{\mu}_0) \cup \\ \cup (\overset{r}{\mu}_0, \overset{l}{\mu}_1) \cup (\overset{r}{\mu}_1, \overset{l}{\mu}_2) \cup \cdots \cup (\overset{r}{\mu}_N, +\infty)\ ,$$

has no zeroes of the generalized order more than $1/2$ in the set

$$\Big((\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \setminus \{\tau_j\}_0^N\Big) \cup \Big((\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \setminus \{-\tau_j\}_0^N\Big),$$

has poles of generalized order greater than or equal to $1/2$ at the points of the set

$$\Big((\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \cap \{\tau_j\}_0^N\Big) \cup \Big((\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \{-\tau_j\}_0^N\Big).$$

Combining Theorem 7.2 with Corollary 3.5 we arrive at the following result.

Corollary 7.3. *Under the conditions* (7.12) *the operator A_{ess} is similar to a selfadjoint operator with absolutely continuous spectrum.*

Proof. Consider the decomposition (6.1) and note that the subspace $\mathfrak{H}_e$ in (6.1) is invariant for the operator A, $\mathfrak{H}_e \in \operatorname{Lat} A$. Alongside the skew decomposition (6.1) we consider the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_e \oplus \mathfrak{H}_e^\perp$. According to this decomposition the characteristic function $\theta_A(\cdot)$ of the operator A admits the factorization $\theta_A(\lambda) = \theta_1(\lambda) \cdot \theta_2(\lambda)$ where $\theta_1(\cdot)$ is the characteristic function of the operator $A_{ess} = A\lceil\mathfrak{H}_e$ and $\theta_2(\cdot)$ is the characteristic function of the operator $A_2 := P_2 A\lceil\mathfrak{H}_e^\perp$, where P_2 is the orthoprojection in $\mathfrak{H}$ onto $\mathfrak{H}_e^\perp$. Note, that $\theta_2(\cdot) = \theta_{A_2}(\cdot)$ is a finite Blaschke product since $\sigma(A_{disc})$ is finite.

It follows from (2.18) that $M_+(\cdot)$ (resp. $M_-(\cdot)$) admits a continuous extension to the real line with exception of the set of (real) zeros $\{s_k\}_1^{N+1}$ (resp. $\{-s_k\}_1^{N+1}$) of the polynomial $S(\lambda)$ (resp. $S(-\lambda)$). Moreover, it is clear from the formula (3.21) for the characteristic function $\theta_A(\lambda)$ that real singularities (resp. poles) of $\theta_A(\lambda)$ coincide with the set of real (resp. non-real) roots of the function

$$F(\lambda) = P(\lambda)Q(-\lambda) + iP(\lambda)\sqrt{R(-\lambda)} - P(-\lambda)Q(\lambda) + iP(-\lambda)\sqrt{R(\lambda)}. \tag{7.13}$$

In particular, the numbers of real singularities and poles of $\theta_A(\cdot)$ are finite.

Note, that $A_2^* = A^*\lceil\mathfrak{H}_\theta^\perp$ and $\theta_2^{-1}(\lambda) = \theta_{A_2^*}(\lambda)$ is a finite Blaschke product too. Therefore the sets of real singularities of functions $\theta(\cdot)$ and $\theta_1(\cdot) = \theta(\cdot) \cdot \theta_2^{-1}(\cdot)$ coincide. In particular, $\theta_1(\cdot)$ may have only finite number of singularities and we can apply Proposition 3.4. Therefore, combining Theorem 7.2 with Proposition 3.4 we obtain that A_{ess} is similar to a selfadjoint operator with absolutely continuous spectrum. $\square$

Corollary 7.4. *Let $L = -d^2/dx^2 + q(x)$ be a nonnegative Sturm-Liouville operator with a finite-zone potential q. Then the operator $A = (\operatorname{sgn} x)L$ is similar to a selfadjoint operator with absolutely continuous spectrum.*

Proof. By (7.6), we have $\operatorname{supp} d\Sigma_{ac\pm}(t) \subset \mathbb{R}_\pm$ and $\operatorname{Im} M_\pm(t) = \pi\Sigma'_{ac\pm}(t)$ for almost all $t \in \mathbb{R}$. Therefore,

$$\frac{|\Sigma'_{ac\pm}(t)|^2}{|M_+(t) - M_-(t)|^2} \le \frac{|\Sigma'_{ac\pm}(t)|^2}{\pi^2|\Sigma'_{ac\pm}(t)|^2 + |\operatorname{Re} M_+(t) - \operatorname{Re} M_-(t)|^2} \le \frac{1}{\pi^2},$$

for almost all $t \in \mathbb{R}$. Thus, by Theorem 7.2, A_{ess} is similar to a selfadjoint operator.

The operator A is J–nonnegative. Besides, L has an absolutely continuous spectrum (see, for example, [38]). Hence, $\ker L = 0$. Combining this with Proposition 2.2, we see that all the eigenvalues of A are real and simple. Therefore, by Theorem 7.1, the operator A_{disc} is similar to a selfadjoint operator. Thus, A is similar to a selfadjoint operator. ☐

7.2 Proof of Theorem 7.2

The implication $(i) \Rightarrow (ii)$ follows from Proposition 6.2.

$(ii) \Rightarrow (iii)$. It follows from (7.7) and (7.12) that there are no generalized zeroes of the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ in the set $\cup_{j=0}^N(\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1})$. Likewise, it follows from (7.7) and (7.12) that there are no generalized zeroes of the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ in the set $\cup_{j=0}^N(-\overset{l}{\mu}_{j+1}, -\overset{r}{\mu}_j)$.

It follows from (7.8), (7.9), and (7.12) that the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ has no zeroes of generalized order greater than 1/2 in the sets

$$(\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \setminus \{\tau_j\}_0^N \quad \text{and} \quad (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \setminus \{-\tau_j\}_0^N.$$

It follows from (7.10), (7.11), (7.12), and (7.12) that all the points of the sets

$$(\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N) \cap \{\tau_j\}_0^N \quad \text{and} \quad (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \{-\tau_j\}_0^N$$

are generalized poles of $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$. The generalized orders of these poles are greater than or equal to 1/2.

$(iii) \Rightarrow (i)$. By Theorem 7.1 (1), Assumption (6.1) is fulfilled for the operator A. It follows from (7.1) that we can apply Theorem 6.3.

Let Statement (iii) be fulfilled. We construct the functions $U_\pm(\lambda)$, $w_\pm(t)$, $U_j^\pm$ and the sets G_d, $D_j^\pm$ such that all the conditions of Theorem 6.3 hold true.

Let G_d be any compact set such that $\sigma_{ess}(A) \cap G_d = \emptyset$ and all the points of the set $\sigma_{disc}(A)$ are interior points of G_d.

The set $\sigma_{disc}(A) \cap \mathbb{C}_+$ is finite. Besides,

$$\sigma_{disc}(A) \cap \mathbb{C}_+ = \{\lambda \in \mathbb{C}_+ \ : \ M_+(\lambda) - M_-(\lambda) = 0\}.$$

Let $B_{\mathbb{C}}(\lambda)$ be a finite Blaschke product (see [18]) with the same zeroes in $\mathbb{C}_+$ as $M_+(\lambda) - M_-(\lambda)$. Then $M_+(\lambda) - M_-(\lambda) = B_{\mathbb{C}}(\lambda)M_1(\lambda)$, the function $M_1(\lambda)$ being holomorphic on $\mathbb{C}_+$. Besides,

$$\frac{1}{M_1(\lambda)} \in H(\mathbb{C}_+), \qquad M_1(\lambda) \asymp (M_+(\lambda) - M_-(\lambda)) \quad (\lambda \in \mathbb{C}_+ \setminus G_d^+), \tag{7.14}$$

where G_d^+ is any compact subset of $G_d \cap \mathbb{C}_+$ such that all the points of the set $\sigma_{disc}(A) \cap \mathbb{C}_+$ are interior points of G_d^+.

The set $\sigma_{disc}(A_2^+) \cap \sigma_{disc}(A_2^-) = \{\theta_j^+\}_1^{N_\theta^+} \cap \{\theta_j^-\}_1^{N_\theta^-}$ is finite. By Theorem 7.1, this set is a subset of $\sigma_{disc}(A)$. Let

$$\{\theta_j\}_{j=1}^{N_\theta} := \sigma_{disc}(A_2^+) \cap \sigma_{disc}(A_2^-), \quad N_\theta < \infty.$$

Each point of the set $\{\theta_j\}_{j=1}^{N_\theta}$ is either a pole of the first order or a removable singularity of the function $M_+(\lambda) - M_-(\lambda)$. By κ_j denote the generalized order of a zero of $M_+(\lambda) - M_-(\lambda)$ at θ_j. Then $\kappa_j \in \{-1, 0\} \cup \mathbb{N}$, $j = 1, \ldots, N_\theta$. By Theorem 7.1, we have

$$\{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm} = \{\theta_j^\pm\}_1^{N_\theta^\pm} \setminus \{\theta_j\}_1^{N_\theta}$$

(the sets $\{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm}$ are defined by (6.9)).

Put

$$\{\widetilde{\theta}_j\}_1^{\widetilde{N}_\theta} = (\mathbb{R} \cap \sigma_{disc}(A)) \setminus \{\theta_j\}_1^{N_\theta} .$$

The functions $M_\pm(\lambda)$ are regular at $\widetilde{\theta}_j$ and $M_+(\widetilde{\theta}_j) - M_-(\widetilde{\theta}_j) = 0$, $j = 1, \ldots, \widetilde{N}_\theta$. Let us denote generalized order of $\widetilde{\theta}_j$ as a zero of $M_+(\lambda) - M_-(\lambda)$ by $\widetilde{\kappa}_j$ (clearly, $\widetilde{\kappa}_j \in N$).

Put $M_2(\lambda) := M_1(\lambda)/B_\theta$, where

$$B_\theta(\lambda) := \frac{\prod_{j=1}^{N_\theta} (\lambda - \theta_j)^{\kappa_j + 1}}{\prod_{j=1}^{N_\theta} (\lambda - (\theta_j - i\varepsilon_1))^{\kappa_j + 1}} \; \frac{\prod_{j=1}^{\widetilde{N}_\theta} (\lambda - \widetilde{\theta}_j)^{\widetilde{\kappa}_j}}{\prod_{j=1}^{\widetilde{N}_\theta} \left(\lambda - (\widetilde{\theta}_j - i\varepsilon_1)\right)^{\widetilde{\kappa}_j}}.$$

Here and below ε_1 is an arbitrary fixed positive number. Taking into account (7.14), we get

$$\frac{1}{M_2(\lambda)} \in H(\mathbb{C}_+), \qquad M_2(\lambda) \asymp (M_+(\lambda) - M_-(\lambda)) \quad (\lambda \in \mathbb{C}_+ \setminus G_d). \tag{7.15}$$

Denote

$$\rho_1 := \rho(L) \cup \rho(-L) .$$

If λ_0 is a generalized zero of $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ and $\lambda_0 \in \rho_1$, then $\lambda_0 \in \{\theta_j\}_{j=1}^{N_\theta} \cup \{\widetilde{\theta}_j\}_{j=1}^{\widetilde{N}_\theta}$. Moreover, it follows from Statement (iii) that the function $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ has no generalized zeroes in the set $\sigma_1 := \sigma_1^+ \cup \sigma_1^-$, where

$$\sigma_1^\pm := \pm \bigcup_{j=0}^{N} (\overset{r}{\mu}_j, \overset{l}{\mu}_{j+1})$$

are the sets of interior points of the spectra $\sigma(\pm L)$. Therefore the definition of B_θ imply that

$$M_2^{-1}(\lambda) = O(1) \quad (\lambda \to \lambda_0) \quad \forall \lambda_0 \in \rho_1 \cup \sigma_1, \tag{7.16}$$

$$M_2(\lambda) \asymp (\theta_j^\pm - \lambda)^{-1} \quad (\lambda \to \theta_j^\pm), \quad j = 1, \ldots, N_\theta^\pm. \tag{7.17}$$

Let us explain formula (7.17). If $\theta_j^\pm \in \{\theta_j\}_1^{N_\theta}$, the formula (7.17) follows from the definition of the function B_θ. If $\theta_j^\pm \in \{\widetilde{\theta}_j^\pm\}_1^{\widetilde{N}_\theta^\pm}$, the asymptotics

$$M_\pm(\lambda) \asymp (\widetilde{\theta}_j^\pm - \lambda)^{-1}, \quad M_\mp(\lambda) = O(1)\, (\widetilde{\theta}_j^\pm - \lambda)^{-1/2} \quad (\lambda \to \widetilde{\theta}_j^\pm)$$

and (7.14) imply (7.17).

Let us denote

$$\{\zeta_j^{\pm}\}_1^{N_\zeta^{\pm}} := \left\{ z \in \{\pm \overset{r}{\mu}_j\}_0^N \cup \{\pm \overset{l}{\mu}_j\}_1^N \;:\; z \text{ is a generalized zero of } \overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda) \right\}.$$

By Statement (iii), the generalized orders of all the zeroes $\zeta_j^{\pm}$ are equal to 1/2. It follows from Statement (iii) and asymptotics for $M_{\pm}(\lambda)$ that

$$\{\zeta_j^{\pm}\} \subset (\{\pm \overset{r}{\mu}_j\}_0^N \cup \{\pm \overset{l}{\mu}_j\}_1^N) \setminus (\{\theta_j^{\mp}\}_1^{N_\theta^{\mp}} \cup \sigma_1). \tag{7.18}$$

Denote

$$\{\zeta_j\}_1^{N_\zeta} := \{\zeta_j^{+}\}_1^{N_\zeta^{+}} \cap \{\zeta_j^{-}\}_1^{N_\zeta^{-}}, \quad \{\widetilde{\zeta_j^{\pm}}_1\}^{\widetilde{N_\zeta^{\pm}}} := \{\zeta_j^{\pm}\}_1^{N_\zeta^{\pm}} \setminus \{\zeta_j\}_1^{N_\zeta}. \tag{7.19}$$

Statement (iii) imply $\widetilde{\zeta_j^{\pm}} \notin \sigma_1$. Besides,

$$\widetilde{\zeta_j^{\pm}} \notin \{\zeta_j\}_1^{N_\zeta} = \{\zeta_j^{\pm}\}_1^{N_\zeta^{\pm}} \cap (\{\mp \overset{r}{\mu}_j\}_0^N \cup \{\mp \overset{l}{\mu}_j\}_1^N),$$

therefore $\widetilde{\zeta_j^{\pm}} \in \rho_1^{\mp}$, where

$$\rho_1^{\pm} := \pm \rho(L) \; (= \pm \bigcup_{j=0}^{N} (\overset{l}{\mu}_j, \overset{r}{\mu}_j)) \;.$$

Put

$$u_{\pm}(\lambda) := \frac{\sqrt{R(\pm\lambda)}}{S(\pm\lambda)} \; \frac{\prod_{j=1}^{N_\theta^{\pm}} (\lambda - \theta_j^{\pm})}{\prod_{j=1}^{N_\theta^{\pm}} (\lambda - (\theta_j^{\pm} - i\varepsilon_1))} \; \frac{\prod_{j=1}^{\widetilde{N}_\zeta^{\pm}} (\lambda - \widetilde{\zeta}_j^{\mp})}{\prod_{j=1}^{\widetilde{N}_\zeta^{\pm}} (\lambda - (\widetilde{\zeta}_j^{\mp} - i\varepsilon_1))}, \tag{7.20}$$

Now we define $U_{\pm}$ by

$$U_{\pm} := \frac{\sqrt{u_{\pm}(\lambda)}}{M_2(\lambda)}.$$

Let us check conditions (6.10), (6.11), and (6.12). All the asymptotics given below are considered on $\overline{\mathbb{C}_+}$, unless otherwise specified.

Lemma 7.5. *Let Statement (iii) be true. Then condition* (6.10) *is fulfilled, i.e.,*

$$\frac{\operatorname{Im} M_{ac\pm}(\lambda)}{|M_+(\lambda) - M_-(\lambda)|^2} \le C_{\pm}^u |U_{\pm}(\lambda)|^2, \quad \lambda \in \mathbb{C}_+ \setminus G_d \;.$$

Proof. By (7.15), condition (6.10) is equivalent to

$$\operatorname{Im} M_{ac\pm}(\lambda) = O(1)\, u_{\pm}(\lambda), \qquad (\lambda \in \overline{\mathbb{C}_+ \setminus G_d}). \tag{7.21}$$

Since

$$M_{ac\pm}(\lambda) \asymp M_{\pm}(\lambda) \asymp |\lambda|^{-1/2} \quad (\lambda \to \infty) \tag{7.22}$$

$$\text{and} \qquad u_\pm(\lambda) \asymp |\lambda|^{-1/2} \quad (\lambda \to \infty), \tag{7.23}$$

$$\text{we have} \qquad \frac{\operatorname{Im} M_{ac\pm}(\lambda)}{u_\pm(\lambda)} = O(1) \quad (\lambda \to \infty).$$

If $\lambda_0 \in \overline{(\mathbb{C}_+ \setminus G_d)} \setminus \left(\{\pm \overset{r}{\mu}_j\}_0^N \cup \{\pm \overset{l}{\mu}_j\}_1^N \cup \{\widetilde{\zeta_j^\mp}\}_1^{\widetilde{N_\zeta^\mp}} \right)$, then

$$\operatorname{Im} M_{ac\pm}(\lambda) = O(1) \quad (\lambda \to \lambda_0), \qquad u_\pm(\lambda) \asymp 1 \quad (\lambda \to \lambda_0).$$

Let $\lambda_0 \in (\{\pm \overset{r}{\mu}_j\}_0^N \cup \{\pm \overset{l}{\mu}_j\}_1^N) \setminus \{\pm \tau_j\}_0^N$. Then (2.18) yields

$$\operatorname{Im} M_{ac\pm}(\lambda) \asymp \operatorname{Im} M_\pm(\lambda) = O(|\lambda - \lambda_0|^{1/2}) \quad (\lambda \to \lambda_0);$$

$$\text{besides,} \qquad u_\pm(\lambda) \asymp |\lambda - \lambda_0|^{1/2} \quad (\lambda \to \lambda_0).$$

Let $\lambda_0 \in (\{\pm \overset{r}{\mu}_j\}_0^N \cup \{\pm \overset{l}{\mu}_j\}_1^N) \cap \{\pm \tau_j\}_0^N$. Then (2.18) yields

$$\operatorname{Im} M_{ac\pm}(\lambda) \asymp \operatorname{Im} M_\pm(\lambda) = O(|\lambda - \lambda_0|^{-1/2}) \quad (\lambda \to \lambda_0);$$

$$\text{besides,} \qquad u_\pm(\lambda) \asymp |\lambda - \lambda_0|^{-1/2} \quad (\lambda \to \lambda_0).$$

Let $\lambda_0 \in \{\widetilde{\zeta_j^\mp}\}_0^{\widetilde{N_\zeta^\mp}}$. Then (7.18) and (2.8) yield

$$\operatorname{Im} M_{ac\pm}(\lambda) \asymp \operatorname{Im} M_\pm(\lambda) = O(\operatorname{Im} \lambda) = O(\lambda - \lambda_0) \quad (\lambda \to \lambda_0)\ .$$

On the other hand,

$$u_\pm(\lambda) \asymp |\lambda - \lambda_0| \quad (\lambda \to \lambda_0).$$

If we combine all these estimates, we get (7.21). Thus (6.10) is proved. □

Lemma 7.6. *Condition* (6.11) *is fulfilled, i.e.,* $U_\pm(\lambda) \in \mathcal{N}^+(C_+)$.

Proof. The functions $U_\pm(\lambda)$ are holomorphic on $\mathbb{C}_+$ by definition. Since

$$M_+(\lambda) - M_-(\lambda) \asymp |\lambda|^{-1/2} \quad (\lambda \to \infty)\ ,$$

(7.23) imply the following formula

$$U_\pm(\lambda) \asymp |\lambda|^{1/4} \qquad (\lambda \to \infty). \tag{7.24}$$

Condition (6.11) follows from (7.24) and Lemma 2.11. □

Lemma 7.7. *Let Statement (iii) be true. Then condition* (6.12) *is fulfilled, i.e.,*

$$\frac{U_\pm(t)}{\theta_j^- - t} \in L^2(\mathbb{R}),\ j = 1, \cdots, N_\theta^-; \quad \frac{U_\pm(t)}{\theta_j^+ - t} \in L^2(\mathbb{R}),\ j = 1, \cdots, N_\theta^+\ .$$

Proof. The definition of the polynomial $S(\lambda)$ imply

$$|u_\pm(\lambda)| = \frac{\prod\limits_{(\pm\lambda_0)\in(\{\overset{r}{\mu}_j\}_0^N\cup\{\overset{l}{\mu}_j\}_1^N)\setminus\{\tau_j\}_0^N} |\lambda-\lambda_0|^{1/2}}{\prod\limits_{(\pm\lambda_0)\in\{\tau_j\}_0^N\cap(\{\overset{r}{\mu}_j\}_0^N\cup\{\overset{l}{\mu}_j\}_1^N)} |\lambda-\lambda_0|^{1/2}\ \prod\limits_{j=1}^{N_\theta^\pm}|\lambda-(\theta_j^\pm - i\varepsilon_1)|}\times$$
$$\times\frac{\prod\limits_{j=1}^{\widetilde{N}_\zeta^\mp}|\lambda-\widetilde{\zeta}_j^\mp|}{\prod\limits_{j=1}^{\widetilde{N}_\zeta^\mp}|t-(\widetilde{\zeta}_j^\mp - i\varepsilon_1)|^{1/2}}. \tag{7.25}$$

It follows from (7.25), (7.16), (7.20), (7.17), Statement (iii), and the definition of $\{\widetilde{\zeta}_j^\mp\}_1^{\widetilde{N}_\zeta^\mp}$ that

$$U_\pm(\lambda) = O(1) \quad (\lambda\to\lambda_0), \quad \lambda_0\in\sigma_1^+\cup\sigma_1^-\cup\rho_1^\pm \tag{7.26}$$
$$U_\pm(\lambda) \asymp (\lambda-\theta_j^\pm) \quad (\lambda\to\theta_j^\pm), \quad j=1,\dots,N_\theta^\pm, \tag{7.27}$$
$$U_\pm(\lambda) = O(1)\,|\lambda-\theta_j^\mp|^{3/4} \quad (\lambda\to\theta_j^\mp), \quad j=1,\dots,N_\theta^\mp, \tag{7.28}$$
$$U_\pm(\lambda) = O(|\lambda-\lambda_0|^{-1/4}) \ \ (\lambda\to\lambda_0), \quad \lambda_0\in\Big(\{\pm\overset{r}{\mu}_j\}_{j=0}^N\cup\{\pm\overset{l}{\mu}_j\}_{j=1}^N\Big)\setminus\{\pm\tau_j\}_{j=0}^N, \tag{7.29}$$
$$U_\pm(\lambda) = O(|\lambda-\lambda_0|^{1/4}) \ \ (\lambda\to\lambda_0), \quad \lambda_0\in\Big(\{\pm\overset{r}{\mu}_j\}_{j=0}^N\cup\{\pm\overset{l}{\mu}_j\}_{j=1}^N\Big)\cap\{\pm\tau_j\}_{j=0}^N. \tag{7.30}$$
$$U_\mp(\lambda) = O(|\lambda-\lambda_0|^{1/4}) \ \ (\lambda\to\lambda_0), \quad \lambda_0\in\Big(\{\pm\overset{r}{\mu}_j\}_{j=0}^N\cup\{\pm\overset{l}{\mu}_j\}_{j=1}^N\Big)\cap\{\pm\tau_j\}_{j=0}^N. \tag{7.31}$$

Therefore, $\dfrac{U_\pm(t)}{\theta_j^+ - t}\in L^2_{loc}(\mathbb{R})$ and $\dfrac{U_\pm(t)}{\theta_j^- - t}\in L^2_{loc}(\mathbb{R})$. Combining this with (7.24), we get (6.12). ☐

Let $w_\pm(t)$ be defined by

$$\frac{1}{w_\pm(t)} := \left|\frac{\sqrt{R(\pm t)}}{S(\pm t)}\ \frac{\prod\limits_{j=1}^{N_\theta^\pm}(t-\theta_j^\pm)}{\prod\limits_{j=1}^{N_\theta^\pm}(t-(\theta_j^\pm - i\varepsilon_1))}\ \frac{\prod\limits_{j=1}^{\widetilde{N}_\zeta^\mp}(\lambda-\widetilde{\zeta}_j^\mp)^{1/2}}{\prod\limits_{j=1}^{\widetilde{N}_\zeta^\mp}(\lambda-(\widetilde{\zeta}_j^\mp - i\varepsilon_1))^{1/2}}\right|.$$

Let us check conditions (6.13), (6.14), and (6.15).

Since all the points $\theta_j^\pm$, $\zeta_j^\mp$ belongs to $\rho_1^\pm(=\mathbb{R}\setminus\operatorname{supp} d\Sigma_{ac\pm})$, formulae (7.7)–(7.11) imply (6.13).

Lemma 7.8. *Condition* (6.14) *is fulfilled, i.e., the weights w_+ and w_- satisfy the* (A_2) *condition.*

We give two proofs of this lemma. The fist proof is based on the Hunt- Muckenhoupt-Wheeden theorem, the second on the Helson-Szegö theorem. Note that [21, Theorem 4] can be used also.

Proof 1 of Lemma 7.8 . It is clear that all the conditions of Proposition 2.9 is fulfilled for the functions $w_\pm$. Thus, $w_\pm \in (A_2)$. $\square$

Proof 2 of Lemma 7.8 . The Helson-Szegö condition (see (2.34)) is equivalent to the (A_2) condition. Let us prove that condition (2.34) is satisfied for w_+.

Obviously,

$$w_+(t) = \frac{\prod\limits_{\lambda_0 \in \{\tau_j\}_0^N \cap (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N)} |t-\lambda_0|^{1/2} \prod\limits_{j=1}^{N_\theta^+} |t-(\theta_j^+ - i\varepsilon_1)| \prod\limits_{j=1}^{\widetilde{N}_\zeta^-} |t-(\widetilde{\zeta}_j^- - i\varepsilon_1)|^{1/2}}{\prod\limits_{\lambda_0 \in (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_0^N) \setminus \{\tau_j\}_0^N} |t-\lambda_0|^{1/2} \prod\limits_{j=1}^{\widetilde{N}_\zeta^-} |t-\widetilde{\zeta}_j^-|^{1/2}}. \tag{7.32}$$

Consequently,

$$\log w_+(t) = \frac{1}{2} \sum_{\lambda_0 \in \{\tau_j\}_0^N \cap (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N)} \log|t-\lambda_0| + \sum_{j=1}^{N_\theta^+} \log|t-(\theta_j^+ - i\varepsilon_1)| +$$

$$+\frac{1}{2} \sum_{j=1}^{\widetilde{N}_\zeta^-} \log|t-(\widetilde{\zeta}_j^- - i\varepsilon_1)| - \frac{1}{2} \sum_{\lambda_0 \in (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_0^N) \setminus \{\tau_j\}_0^N} \log|t-\lambda_0| -$$

$$-\frac{1}{2} \sum_{j=1}^{\widetilde{N}_\zeta^-} \log|t-\widetilde{\zeta}_j^-| = (\mathcal{H}v_+)(t) + c_1, \tag{7.33}$$

where $\mathcal{H}$ is the Hilbert transform (see Subsection 2.6),

$$v_+(t) = \frac{1}{2} \sum_{\lambda_0 \in (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_0^N) \setminus \{\tau_j\}_0^N} \arg(t-\lambda_0) + \frac{1}{2} \sum_{j=1}^{\widetilde{N}_\zeta^-} \arg(t-\widetilde{\zeta}_j^-) -$$

$$-\frac{1}{2} \sum_{\lambda_0 \in \{\tau_j\}_0^N \cap (\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N)} \arg(t-\lambda_0) - \sum_{j=1}^{N_\theta^+} \arg(t-(\theta_j^+ - i\varepsilon_1)) -$$

$$-\frac{1}{2} \sum_{j=1}^{\widetilde{N}_\zeta^-} \arg(t-(\widetilde{\zeta}_j^- - i\varepsilon_1));$$

here c_1 is a constant, the branch of $\arg z$ is fixed by $\arg z \in (-\pi, \pi]$, $z \in \mathbb{C}$.

The function v_+ is bounded and piecewise smooth; the set of jumps of v_+ is

$$\{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N \cup \{\widetilde{\zeta_j^-}\}_1^{\widetilde{N_\zeta^-}}.$$

The absolute values of all the jumps are equal to $\pi/2$. Moreover,

$$v_+(t) \asymp \arctan\frac{1}{t} \asymp \frac{1}{t} \quad (t \to +\infty),$$

$$v_+(t) + \frac{\pi}{2} \asymp \arctan\frac{1}{|t|} \asymp \frac{1}{|t|} \quad (t \to -\infty),$$

$v_+(t)$ monotonically increases on $t \in (-\infty, \overset{r}{\mu}_0)$ and $(\overset{r}{\mu}_N, +\infty)$. Therefore, v_+ can be represented in the form

$$v_+(t) = v_1(t) + v_2(t) - \pi/4,$$

where v_1 is a piecewise continuous function such that

$$\|v_1(t)\|_{L^\infty} < \pi/2, \tag{7.34}$$

$$v_1 \text{ has jumps at the points} \quad \{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N \cup \{\widetilde{\zeta_j^+}\}_1^{\widetilde{N_\zeta^+}},$$

v_2 is a C^1 function on $\mathbb{R}$ such that

$$v_2(t) = 0 \quad \text{for } t \notin [\overset{r}{\mu}_0 - \delta_2, \overset{r}{\mu}_N + \delta_2]; \tag{7.35}$$

here δ_2 is a specified positive number.

From (7.35), we get

$$(Hv_2)(t) \asymp |t|^{-1} \quad (|t| \to \infty)\ .$$

It follows from $v_2 \in C^1(\mathbb{R})$ that $v_2 \in \mathrm{Lip}^\alpha(\mathcal{I})$ for any compact interval $\mathcal{I} \subset \mathbb{R}$ and for any $\alpha \in (0,1)$. If we combine this with Privalov's theorem (see [36]) and (7.35), we get $Hv_2 \in \mathrm{Lip}^\alpha(\mathcal{I})$, $0 < \alpha < 1$. Hence, Hv_2 is a continuous function on $\mathbb{R}$ and (7.35) imply $Hv_2 \in L^\infty(\mathbb{R})$. Taking into account (7.33), we get $\log w_+(t) = (Hv_1)(t) + (Hv_2)(t) + c_1$, where $\|v_1\|_{L^\infty} < \pi/2$, $Hv_2 + c_1 \in L^\infty(\mathbb{R})$. That is w_+ satisfy the Helson-Szegö condition. The condition (6.14) is proved for w_+. In the same way we prove (6.14) for w_-. ☐

Lemma 7.9. *Let Statement (iii) be true. Then condition* (6.15) *is fulfilled, i.e.,*

$$\frac{U_+^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R}), \qquad \frac{U_-^2(t)}{w_\pm(t)} \in L^\infty(\mathbb{R}).$$

Proof. Note that

$$w_+^{-1}(t) \asymp |t|^{-1/2} \quad (|t| \to \infty). \tag{7.36}$$

It follows from (7.24), (7.26)-(7.31), (7.32), Statement (iii), and (7.15) that

$$U_+^2(t) w_+^{-1}(t) \in L^\infty(\mathbb{R}) \quad \text{and}$$

$$U_-^2(t) w_+^{-1}(t) = O(1) \quad (t \to t_0)$$
$$\text{for} \quad t_0 \in \{-\infty\} \cup \{+\infty\} \cup \rho_1^- \cup \sigma_1^- \cup \sigma_1^+ \cup \{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N. \tag{7.37}$$

Note that

$$\mathbb{R} \setminus \left(\rho_1^- \cup \sigma_1^- \cup \sigma_1^+ \cup \{\overset{r}{\mu}_j\}_0^N \cup \{\overset{l}{\mu}_j\}_1^N\right) = (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \rho_1^+. \tag{7.38}$$

If λ_0 is a generalized zero of $\overset{*}{M}_+(\lambda) - \overset{*}{M}_-(\lambda)$ and $\lambda_0 \in (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \rho_1^+$, the definition of the set $\{\widetilde{\zeta_j^-}\}_1^{\widetilde{N_\zeta^-}}$ imply that $\lambda_0 \in \{\widetilde{\zeta_j^-}\}_1^{\widetilde{N_\zeta^-}}$ and the generalized order of λ_0 equals $1/2$. Thus, by (7.15) and the definitions of w_+, U_-, we have

$$U_-^2(t) w_+^{-1}(t) = O(1) \quad (t \to t_0), \quad t_0 \in (\{-\overset{r}{\mu}_j\}_0^N \cup \{-\overset{l}{\mu}_j\}_1^N) \cap \rho_1^+ \ .$$

Taking into account (7.37) and (7.38), we get

$$U_-^2(t) w_+^{-1}(t) \in L^\infty(\mathbb{R}).$$

One can prove $U_\pm^2(t) w_-^{-1}(t) \in L^\infty(\mathbb{R})$ in the same way. Thus (6.15) is proved. ☐

Let $\widetilde{\theta}_j^\pm$ be a point of the set $\{\widetilde{\theta}_k^\pm\}_1^{\widetilde{N}_\theta^\pm}$. Let $D_j^\pm$ be a sufficiently small neighborhood of $\widetilde{\theta}_j^\pm$ such that

$$D_j^\pm \cap \left(\{\theta_k^\pm\}_1^{N_\theta^\pm} \cup \{\pm \overset{r}{\mu}_k\}_0^N \cup \{\pm \overset{l}{\mu}_k\}_1^N \right) = \widetilde{\theta}_j^\pm.$$

Put

$$U_\theta^\pm := \frac{\sqrt{u_\theta^\pm(\lambda)}}{M_2(\lambda)}, \qquad \text{where} \qquad u_\theta^\pm(\lambda) := \frac{\sqrt{R(\pm\lambda)}}{S(\pm\lambda)} \, \frac{\prod_{j=1}^{\widetilde{N}_\zeta^\pm} (\lambda - \widetilde{\zeta}_j^\mp)}{\prod_{j=1}^{\widetilde{N}_\zeta^\pm} (\lambda - (\widetilde{\zeta}_j^\mp - i\varepsilon_1))}. \tag{7.39}$$

We define $U_j^\pm$ as $U_j^\pm := U_\theta^\pm$ for all $j = 1, \ldots, \widetilde{N}_\theta^\pm$.

Lemma 2.11 imply that $U_\theta^\pm \in \mathcal{N}^+(\mathbb{C}_+)$.

Lemma 7.10. *Let Statement (iii) be true. Then conditions* (6.16), (6.17), *and* (6.18) *are fulfilled. That is, for every* $\widetilde{\theta}_j^\pm \in \{\widetilde{\theta}_k^\pm\}_1^{\widetilde{N}_\theta^\pm}$, *the following conditions hold:*

$$\frac{1}{|M_+(\lambda) - M_-(\lambda)|^2} \operatorname{Im} \frac{1}{\widetilde{\theta}_j^\pm - \lambda} \le C_\theta^u |U_\theta^\pm(\lambda)|^2 \quad \textit{for } \lambda \in D_j^\pm \cap \mathbb{C}_+ \ ,$$

$$\frac{|U_\theta^\pm(t)|^2}{w_+(t)} \in L^\infty(\mathbb{R}), \quad \frac{|U_\theta^\pm(t)|^2}{w_-(t)} \in L^\infty(\mathbb{R}) \ ,$$

$$\frac{1}{|\widetilde{\theta}_j^\pm - \lambda| \, |M_+(\lambda) - M_-(\lambda)|} \le C_\theta^M \qquad \textit{for } \lambda \in D_j^\pm \cap \mathbb{C}_+,$$

where C_θ^u *and* C_θ^M *are constants.*

Proof. Note that

$$M_2(\lambda) \asymp M_+(\lambda) - M_-(\lambda) \quad (\lambda \to \widetilde{\theta}_j^\pm).$$

Therefore (6.16) is equivalent to

$$\operatorname{Im} \frac{1}{\widetilde{\theta}_j^\pm - \lambda} \le C_1 |u_\theta^\pm(\lambda)| \quad \text{for } \lambda \in D_j^\pm \cap \mathbb{C}_+ \ . \tag{7.40}$$

By (7.18) and (7.19), it follows that $\widetilde{\theta}_j^{\pm} \notin \{\widetilde{\zeta}_k^{\mp}\}_1^{\widetilde{N}_\zeta^{\pm}}$. Taking into account (7.39) and (7.3), we see that $u_\theta^{\pm}(\lambda)$ has a pole of the first order at $\widetilde{\theta}_j^{\pm}$. This implies (7.40). Thus (6.16) is proved.

Lemma (7.9) and the definitions of $u^{\pm}$ and $u_\theta^{\pm}$ imply that

$$\frac{|U_\theta^+(t)|^2}{|w_\pm(t)|} \le C_2 \quad \text{for } t \in \mathbb{R} \setminus \bigcup_{k=1}^{\widetilde{N}_\theta^+} D_k^+ \ . \tag{7.41}$$

Hence, to check condition (6.17) for U_θ^+, it is suffices to show that

$$\frac{|U_\theta^+(t)|^2}{|w_\pm(t)|} \le C_2 \quad \text{for } t \in D_k^+, \quad k = 1, \ldots, \widetilde{N}_\theta^+ \ . \tag{7.42}$$

It is easy to see that

$$M_2(\lambda) \asymp (\lambda - \widetilde{\theta}_k^{\pm})^{-1}, \quad u_\theta^{\pm}(\lambda) \asymp (\lambda - \widetilde{\theta}_k^{\pm})^{-1} \quad (t \to \widetilde{\theta}_k^{\pm}) \tag{7.43}$$

$$\frac{1}{w_\pm(t)} = O(1)\,(t - \widetilde{\theta}_k^+)^{1/2} \quad (t \to \widetilde{\theta}_k^+) \ .$$

Combining these formulae, we obtain (7.42). Thus (6.17) for U_θ^+ is proved. The proof of (6.17) for U_θ^- is similar.

Condition (6.18) follows from (7.43). □

Since all the conditions of Theorem 6.3 are fulfilled, we see that A_{ess} is similar to a selfadjoint operator. Theorem 6.3 is proved.

7.3 Examples

Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with a finite-zone potential q. Put

$$A := JL = (\operatorname{sgn} x)(-d^2/dx^2 + q(x)) \ .$$

Definition 7.1. We shall say that a point $a \in \sigma_{ess}(A) \cup \infty$ is *a strong spectral singularity of* A_{ess} if at least one of the following two functions

$$\frac{\Sigma'_{ac+}(t)}{M_+(t) - M_-(t)}, \quad \frac{\Sigma'_{ac-}(t)}{M_+(t) - M_-(t)}$$

is not essentially bounded in any neighborhood of a.

By Theorem 7.2, A_{ess} is similar to a selfadjoint operator if and only if A_{ess} has no strong spectral singularities. Combining Theorems 7.2 and 7.1, we see that A is similar to a normal operator if and only if the following two conditions hold:

1) A_{ess} is similar to a selfadjoint operator;

2) all eigenvalues of A_{disc} are simple.

By $L(\xi, q)$ we denote the Sturm-Liouville operator with a finite-zone potential $q(x) + \xi$,

$$L(\xi, q) := -d^2/dx^2 + q(x) + \xi,$$

where ξ is a real constant. Put

$$A(\xi, q) := JL(\xi, q) = (\operatorname{sgn} x)(-d^2/dx^2 + q(x) + \xi) \ .$$

Let $A_{ess}(\xi, q_1)$ be the part of $A(\xi, q_1)$ on $\mathfrak{H}_e$.

Example 7.1. Consider the following periodic one-zone potential

$$q_1(x) = (1-k^2)(2\,\mathrm{sn}^2(x,k') - 1), \quad k \in (0,1), \quad k' = \sqrt{1-k^2}, \tag{7.44}$$

where $\mathrm{sn}(x,k')$ is the Jacobi elliptic function. Then $L(\xi, q_1)$ is a one-zone periodic operator; $L(\xi, q_1)$ has the gaps $(-\infty, \xi)$ and $(k^2+\xi, 1+\xi)$.

The corresponding Weyl functions $M_\pm(\lambda)$ has the forms

$$M_+(\lambda) = -M_-(-\lambda) = i\frac{\lambda - (\xi+1)}{\sqrt{(\lambda-\xi)(\lambda-(\xi+k^2))}}\ , \quad 0 < k^2 < 1,$$

(see [2, Appendix II]). Theorem 7.2 imply that $A_{ess}(\xi, q_1)$ is similar to a selfadjoint operator if and only if

$$\xi \in [-1, -k^2] \cup [0, \infty).$$

Note that for $\xi \in [-1, -k^2]$ the operator $L(\xi, q_1)$ is not nonnegative. If

$$\xi \in (-1+\sqrt{1-k^2}, -1-\sqrt{1-k^2}),$$

then $A(\xi, q_1)$ has exactly two eigenvalues

$$\pm\sqrt{(\xi+1)^2 - (1-k^2)}\ ;$$

these eigenvalues are simple and nonreal. For sufficiently small $\xi \geq 0$, the potential $q_1(x) + \xi$ is not nonnegative, although $L(\xi, q_1) \geq 0$.

Spectral properties of $A(\xi, q_1)$ are given in more detail in the following table. The abbreviations 'S-A' ('Norm') in the column 'Similarity' means that $A(\xi, q_1)$ is similar to a selfadjoint (normal) operator. 'NonSim' in the column 'Similarity' means that $A(\xi, q_1)$ is not similar to a normal operator. We put $\lambda_\pm(\xi) := \pm\sqrt{(\xi+1)^2 - (1-k^2)}$.

Spectral properties of the operator $A(\xi, q_1)$

Intervals	Strong spectral singularities	Eigenvalues	Similarity
$\xi \in [0, +\infty)$	No	$\lambda_\pm(\xi)$	S-A
$\xi \in (-\frac{k^2}{2}, 0)$	0	$\lambda_\pm(\xi)$	NonSim
$\xi = -\frac{k^2}{2}$	0	No	NonSim
$\xi \in (-1+\sqrt{1-k^2}, -\frac{k^2}{2})$	0, $\lambda_\pm(\xi)$	No	NonSim
$\xi = -1+\sqrt{1-k^2}$	0	No	NonSim
$\xi \in (-k^2, -1+\sqrt{1-k^2})$	0	$\lambda_\pm(\xi)$	NonSim
$\xi \in [-1, -k^2]$	No	$\lambda_\pm(\xi)$	Norm
$\xi \in (-1-\sqrt{1-k^2}, -1)$	0	$\lambda_\pm(\xi)$	NonSim
$\xi \in -1-\sqrt{1-k^2}$	0	No	NonSim
$\xi \in (-\infty, -1-\sqrt{1-k^2})$	0, $\lambda_\pm(\xi)$	No	NonSim

Table 7.1

Remark 7.1. Example 7.1 shows that condition (5.21) is not necessary for similarity of A to a self-adjoint operator. Let us explain this.

Let $\xi > 0$. Then $A(\xi, q_1)$ is similar to selfadjoint operator, but the function $\dfrac{M_+(\lambda) + M_-(\lambda)}{M_+(\lambda) - M_-(\lambda)}$, $\lambda \in \mathbb{C}_+$, is unbounded in neighborhoods of the eigenvalues $\lambda_\pm := \pm\sqrt{(\xi+1)^2 - (1-k^2)}$. Indeed, the functions $M_\pm$ are holomorphic in points $\lambda_\pm$ and $\lambda_\pm$ are zeroes of $M_+(\cdot) - M_-(\cdot)$. On the other hand, it is easy to check that

$$M_+(\lambda_+) < 0, \quad M_-(\lambda_+) < 0, \quad M_+(\lambda_-) > 0, \quad M_-(\lambda_-) > 0.$$

Therefore, $M_+(\lambda_\pm) + M_-(\lambda_\pm) \neq 0$.

Example 7.2. Consider even periodic potential

$$q_2 = -2k^2\big(1 - (1-k^2)\,\mathrm{sn}^2(x, k')\big)^{-1} + 1 + k^2, \qquad k \in (0,1), \quad k' = \sqrt{1-k^2}.$$

The operator $L(\xi, q_2)$ is a one-zone operator with gaps $(-\infty, \xi)$ and $(k^2+\xi, 1+\xi)$. The corresponding Weyl functions $M_\pm(\lambda)$ have the forms (see [2, Appendix II])

$$M_+(\lambda) = -M_-(\lambda) = i\frac{\lambda - (\xi + k^2)}{\sqrt{(\lambda-\xi)(\lambda-(\xi+1))}}\ , \quad 0 < k^2 < 1.$$

The operator $A(\xi, q_2)$ has no eigenvalues for all $\xi \in \mathbb{R}$. Hence, $A_{ess}(\xi, q_2) = A(\xi, q_2)$.

Let $0 < k^2 \leq \frac{1}{2}$. Using Theorem 7.2, we get the following result: The operator $A(\xi, q_2)$ is similar to a selfadjoint operator if and only if $\xi \in [-\frac{1}{2}, -k^2] \cup [0, \infty)$. The following table describes spectral properties of $A(\xi, q_2)$.

Spectral properties of $A(\xi, q_2)$, the case $k^2 \in (\,0, 1/2\,]$

Intervals ξ	Strong spectral singularities	Similarity
$\xi \in [0, +\infty)$	No	S-A
$\xi \in (-k^2, 0)$	0	NonSim
$\xi \in [-\frac{1}{2}, -k^2]$	No	S-A
$\xi \in [-1, -\frac{1}{2})$	$\pm\sqrt{(\xi+k^2)^2 + k^2(1-k^2)}$	NonSim
$\xi \in (-\infty, -1)$	$0, \pm\sqrt{(\xi+k^2)^2 + k^2(1-k^2)}$	NonSim

Table 7.2

Assume $k^2 > \frac{1}{2}$. Then $A(\xi, q_2)$ is similar to a selfadjoint operator if and only if $\xi \geq 0$. That is $A(\xi, q_2)$ is similar to a selfadjoint operator iff $L(\xi, q_2) \geq 0$. The following table gives a description of spectral properties of $A(\xi, q_2)$ in this case.

Spectral properties of $A(\xi, q_2)$, the case $k^2 \in (\,1/2, 1\,)$

Intervals	Strong spectral singularities	Similarity
$\xi \in [0, +\infty)$	No	S-A
$\xi \in [-\frac{1}{2}, 0)$	0	NonSim
$\xi \in (-k^2, -\frac{1}{2})$	$0, \pm\sqrt{(\xi+k^2)^2 + k^2(1-k^2)}$	NonSim
$\xi \in [-1, -k^2]$	$\pm\sqrt{(\xi+k^2)^2 + k^2(1-k^2)}$	NonSim
$\xi \in (-\infty, -1)$	$0, \pm\sqrt{(\xi+k^2)^2 + k^2(1-k^2)}$	NonSim

Table 7.3

Example 7.3. Let q_3 be potential (7.44) with $k^2 = 1/2$. Let $\xi \in [-1, -1/2)$. Then, combining Example 7.1 with Theorem 2.1, we see that $A(\xi, q_3)$ is not definitizable, although $A_{ess}(\xi, q_3)$ is similar to a selfadjoint operator and $A(\xi, q_3)$ is similar to a normal operator. The nonreal spectrum of $A(\xi, q_3)$ consists of two simple eigenvalues $\lambda_{\pm}(\xi) := \pm\sqrt{(\xi+1)^2 - (1-k^2)}$. The operator $A(\xi, q_3)$ has no real eigenvalues.

References

[1] N.I. Akhiezer, I.M. Glazman, Theory of linear operators in Hilbert space, Dover, New York, 1993.

[2] N.I. Akhieser, I.M. Glasmann, Theory of linear operators in Hilbert space, Part II, Vischa Shkola, Kharkov, 1978 [Russian].

[3] T.Ya. Azizov, I.S. Iokhvidov, Linear operators in spaces with an indefinite metric, John Wiley and Sons, Chichester, New York, 1989.

[4] R. Beals, Indefinite Sturm-Liouville problems and Half-range completeness, J. Differential Equations **56** (1985), 391–407.

[5] M.S. Brodskii, Triangular and Jordan Representations of Linear Operators, Providence, R.I., AMS, 1971.

[6] J.A. van Casteren, Operators similar to unitary or selfadjiont ones, Pacific J. Math. **104** (1983) no. 1, 241–255.

[7] B. Ćurgus, H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, J. Differential Equations 79 (1989), 31–61.

[8] B. Ćurgus, B. Najman, The operator $(\operatorname{sgn} x)\frac{d^2}{dx^2}$ is similar to a selfadjoint operator in $L^2(\mathbb{R}^n)$, Proc. Amer. Math. Soc. **123** (1995), 1125–1128.

[9] B. Ćurgus, B. Najman, Positive differential operators in Krein space $L^2(\mathbb{R})$, Recent development in operator theory and its applications (Winnipeg, MB, 1994), Oper. Theory Adv. Appl., Birkhäuser, Basel, **87** (1996), 95–104.

[10] B. Ćurgus, B. Najman, Positive differential operators in the Krein space $L^2(\mathbb{R}^n)$, Oper. Theory Adv. Appl., Birkhäuser, Basel. **106** (1998), 113-130.

[11] V.A. Derkach, M.M. Malamud, On the Weyl function and Hermitian operators with gaps, Dokl. AN SSSR **293** (1987), 1041–1046.

[12] V.A. Derkach, M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. **95** (1991), 1–95.

[13] V.A. Derkach, M.M. Malamud, Characteristic functions of almost solvable extensions of Hermitian operators, Ukrainian Math. J., **44** (1992), 435–459.

[14] V.A. Derkach, M.M. Malamud, The extension theory of hermitian operators and the moment problem, J. of Math. Sciences. **73** (1995) no. 2, 141–242.

[15] M.M. Faddeev, R.G. Shterenberg, On similarity of singular differential operators to a selfadjoint one, Zapiski Nauchnyh Seminarov POMI **270**, Issledovaniya po Lineinym Operatoram i Teorii Funktsii **28** (2000), 336–349.

[16] A. Fleige, A spectral theory of indefinite Krein-Feller differential operators, Mahematical Research **98**, Akademie Verlag, Berlin, 1996.

[17] Fleige, A., Najman, B., Nonsingularity of critical points of some differential and difference operators, Oper. Theory: Adv. Appl., **102**, Birkhäuser, Basel, 1998.

[18] J.B. Garnett, Bounded analytic functions, Academic Press, New York, London, 1981.

[19] V.I. Gorbachuk, M.L. Gorbachuk, Boundary value problems for operator differential equations, Mathematics and Its Applications, Soviet Series 48, Dordrecht ets., Kluwer Academic Publishers, 1991.

[20] A.M. Gomilko, On conditions on the generator of uniformly bounded C_0-semigroup of operators, Func. Anal. Appl., **33** (1999) no. 4, 66-69.

[21] H. Helson, G. Szegö, A problem in prediction theory, Ann. Mat. Pura Appl, **51** (1960), 107–138.

[22] P. Jonas, H. Langer, Compact perturbations of definitizable operators, J. Operator Theory **2** (1979), 63–77.

[23] P. Jonas, C. Trunk, On a class of analytic operator functions and their linearizations, Math. Nachrichten **243** (2002), 92–133.

[24] I.S. Kac, M.G. Krein, R-functions — analitic functions mapping the upper halfplane into itself, Supplement to the Russian edition of F.V. Atkinson, Discrete and conyinuous boundary problems, Mir, Moscow, 1968 (Russian). Engl. transl.: Amer. Math. Soc. Transl. Ser. 2, **103** (1974), 1–18.

[25] V. Kapustin, Spectral analysis of almost unitary operators, St. Petersburg Math. J. **13** (2002) no. 5, 739–756.

[26] H.G. Kaper,M.K. Kwong, C.G. Lekkerkerker, A. Zettl, Full- and partial- range expansions for Sturm-Liouville problems with indefinite weights, Proc. Roy. Soc. Edinburgh Sect. A **98** (1984), 69-88.

[27] I.M. Karabash, The operator $-(\operatorname{sgn} x)\frac{d^2}{dx^2}$ is similar to a selfadjoint operator in $L^2(\mathbb{R})$, Spectral and Evolution problems, Proc. of the Eighth Crimean Autumn Math. School-Symposium, Simferopol **8** (1998), 23–26.

[28] I.M. Karabash, J-selfadjoint ordinary differential operators similar to selfadjoint operators, Methods of Functional Analysis and Topology **6** (2000) no. 2, 22–49.

[29] I.M. Karabash, On J-selfadjoint differential operators similar to selfadjoint operators, Math. Notes **68** (2000) no. 6, 798–799.

[30] I.M. Karabash, On differential operators of the first order nonsimilar to selfadjoint ones, Spectral and Evolution problems, Proc. of the Tenth Crimean Autumn Math. School-Symposium, Simferopol **10** (2000), 22–25.

[31] I.M. Karabash, On similarity of differential operators to selfadjoint ones, Candidate thesis, The Institute of Applied Mathemtics and Mechanics NASU, Donetsk, 2005 [Russian].

[32] I.M. Karabash, On eigenvalues in the essential spectrum of Sturm-Liouville operators with the indefinite weight sgn x, Spectral and Evolution problems, Proc. of the Fifteenth Crimean Autumn Math. School-Symposium, Simferopol **15** (2005) 55–60.

[33] I.M. Karabash, M.M. Malamud, The similarity of a J-self-adjoint Sturm-Liouville operator with finite-gap potential to a self-adjoint operator, Doklady Mathematics **69** (2004) no. 2, 195–199.

[34] T. Kato, Perturbation theory for linear operators, Springer-Verlag, Berlin, Heidelberg, New York, 1966.

[35] A.V. Kiselev, M.M. Faddeev, On a problem of similarity for nonselfadjoint operarors with absolutely continuous spectrum, Func. Anal. **34** (2001) no. 2, 78–81.

[36] P. Koosis, Introduction to H_p spaces, Cambridge University Press, Cambrige, 1980.

[37] H. Langer, Spectral functions of definitizable operators in Krein space, Lecture Notes in Mathematics **948** (1982), 1–46.

[38] B.M. Levitan, Inverse Sturm-Liouville problems, Nauka, Moscow 1984 [Russian]; Engl. transl.: VNU Science Press, Utrecht, 1987.

[39] B.M. Levitan, I.S. Sargsjan, Sturm-Liouville and Dirac operators, Nauka, Moscow, 1988 [Russian]; Engl. transl.: Kluwer, Dordrecht 1990.

[40] M.M. Malamud, A criterion for similarity of a closed operator to a self-adjoint one, Ukrainian Math. J. **37** (1985), 49–56 [Russian].

[41] M. M. Malamud, Similarity of a Triangular Operator to a Diagonal Operator, J. Math. Sciences **115** (2000), 2199–2222.

[42] M.M. Malamud, S.M. Malamud, Spectral theory of operator measures in Hilbert spaces, Algebra i Analiz **15** (2003) no. 3, 1–77. (Russian); English translation: St. Petersburg Math. J. **15** (2003) no. 3, 1–53.

[43] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. **165** (1972), 207-226.

[44] R. Hunt, B. Muckenhoupt, R.L. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. **176** (1973) 227-251.

[45] S.N. Naboko, On some conditions of similarity to unitary and selfadjoint operators, Funktsional. Anal. i Prilozhen **18** (1984) no. 1, 16–27 [Russian].

[46] M.A. Naimark, Linear Differential Operators, Part I, Frederic Ungar Publ., New York, 1967.

[47] M.A. Naimark, Linear differential operators,Part II. Ungar, New York, 1968.

[48] N.K. Nikol'skii, Treatise on the Shift Operator, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1986.

[49] N. Nikol'skii, S. Treil, Linear resolvent growth of rank one perturbation of a unitary operator does not imply its similarity to a normal operator, J. D'Analyse Math. **87** (2002), 415–431.

[50] A.I. Parfyonov, On an embedding criterion for interpolation spaces and its applications to indefinite spectral problems, Sib. Mat. Zurnal. **44** (2003) no.4, 810–819 [Russian].

[51] V.P. Potapov, Multiplicative structure of J-nonexpancive matrix functions, Trudy Mosk. Math. Ob. **4** (1955) 125–236 [Russian].

[52] S.G. Pyatkov, Some properties of eigenfunctions of linear pencils, Siberian Math. J. **30** (1989), 587–597.

[53] M. Reed, B. Simon, Methods of modern mathematical physics. I, Academic Press, New York, London, 1980.

[54] M. Reed, B. Simon, Methods of modern mathematical physics. II, Academic Press, New York, London, 1975.

[55] L.A. Sakhnovich, Nonunitary operators with absolutely continuous spectrum on the unit circle, Izv. Acad. Sci. USSR, Ser. Math., **33** (1969) no. 1, 52–64.

[56] A.A. Shkalikov, The indefinite Sturm-Liouville problem: the known and the unknown, Russian Mathematical Surveys **48** (1993), 225–226.

[57] B. Sz.-Nagy, C. Foias, Harmonic analysis of operators on Hilbert space, Amsterdam–Budapest, 1970.

[58] H. Volkmer, Sturm-Liouville problems with indefinite weights and Everitt's inequality, Proc. Roy. Soc. Edinburg **126A** (1996), 1097–1112.