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PERIODIC SOLITARY WAVES IN ONE-DIMENSIONAL HYPERBOLIC STRUCTURES

Introduction. We shall search solitary waves in one-dimensional nonlinear systems described by a scalar hyperbolic equation of the form

$$u_{tt} = u_{xx} + f(u, u_x), \tag{1}$$

with particular concern for the existence of periodic ones.

The method we shall use consists in reducing the problem to ordinary differential equations of second order, and then applying some known results from the theory of plane dynamical systems. In particular, the existence of a periodic solution will be derived by means of Liénard's result on nonlinear oscillations.

The case of parabolic equations of a similar form to (1) has been treated by the same method in a preceding paper of the author [1].

A recent pertinent reference for the concept of solitary wave is M. Remoissenet [6]. For the applications this concept has generated we send the reader to the proceedings volume of the Seventh Kyoto Summer Institute [3] (S. Takeno, Editor). Many other references are indicated in those mentioned above.

V. Mañosa [4] has recently dealt with periodic traveling waves in reaction-diffusion equations, using the Hopf bifurcation technique.

1. The Differential Equation of Solitary Waves. Let us look for solitary waves given by the single variable function

$$y = y(x + ct), \tag{2}$$

where $c \in R$ has to belong to certain intervals, to be determined below.

Substituting $u = y(x + ct)$ in (1), one obtains the ordinary differential equation

$$(c^2 - 1)y'' = f(y, y'), \tag{3}$$

which represents a second order equation any time $c^2 - 1 \neq 0$. The cases $c = \pm 1$ lead to a first order implicit equation. We shall discuss later this situation, when the wave length of the searched solitary wave is the same as in case of the classical equation $u_{tt} = u_{xx}$, i.e.,

$$y = y(x \pm t). \tag{4}$$

The equation (3), for $c^2 - 1 \neq 0$, is a second order autonomous differential equation. The existence of periodic solutions has been amply discussed in the mathematical and applied science literature. For classical results, including existence of limit cycles and related topics, we send the reader to the monographs/treatises authored by E. A. Coddington and N. Levinson [5], G. Sansome and R. Conti [6], G. Birkhoff and G. C. Rota [7], F. Brauer and J. A. Nohel [8].

A special case of (1), which is particularly interesting for us in this paper, corresponds to the choice

$$f(y, y') = g(y)y' + h(y). \tag{5}$$

Some regularity conditions on the functions $g(y)$ and $h(y)$ will be specified below, in accordance with the requirements of the results to be applied in order to derive existence of periodic solutions to the equations (3) or its special case (5).

It is important to point out the fact that, for (3), we need periodic solutions, if we want the solitary waves like (2) to be periodic in time. Moreover, such solutions must be continuous.

For instance, assuming $c^2 \neq 1$, and choosing $f(y, y') = yy'$ product, one obtains solutions to (3) of the form

$$y = \mathrm{tg}\,(C_1 x + C_2), \tag{6}$$

with $C_1 = [2(c^2 - 1)]^{-1}$, and arbitrary C_2. These solutions are periodic, but not continuous (of course, eliminating the constants as trivial solutions).

We shall make one more remark in the case $c^2 - 1 = 0$. In this case, all it remains from the equation (3) is the first order implicit equation

$$f(y, y') = 0. \tag{7}$$

If (7) admits a C^2- solution $y_0(x)$, defined on the whole real axis, then $y_0(x \pm t)$ represents solitary waves for the system described by the equation (1). For instance, for $f(y, y') = y^2 + (y')^2 - 1$, the functions $\cos\,(x \pm t)$ and $\sin\,(x \pm t)$ represent solitary waves for (1).

One can see from the above discussion that equations of the form (1) possess solutions that generate solitary waves.

2. The Equation $u_{tt} = u_{xx} + g(u)u_x + h(u)$. The above equation, which corresponds to the choice of $f(u, u_x)$ given by (5), is quasilinear in u. It turns out to be an appropriate example when equation (3) takes the form which allows us to use a Liénard's type theorem for the existence of a nontrivial periodic solution.

Let us write the equation (3), in case of the special choice for f given by (5):

$$(1 - c^2)y'' + g(y)y' + h(y) = 0. \tag{8}$$

In order to formulate the conditions on $g(y)$ and $h(y)$, required by the use of Liénard's type theorem, we have to distinguish between the cases $|c| < 1$ and $|c| > 1$. We shall deal only with the case $|c| < 1$, leaving to the reader the task to formulate the corresponding conditions in the case $|c| > 1$.

Before we formulate the conditions for equation (8), such that it will possess a periodic solution, let us state here the auxiliary result of Liénard's type we shall rely upon. We take it from [8, Theorem 6.2].

Theorem 1. Consider the equation

$$y'' + a(y)y' + b(y) = 0, \tag{9}$$

under the following assumptions:

(i) $a(y)$ and $b(y)$ are continuously differentiable maps from R into R;

(ii) $yb(y) > 0$ for $y \neq 0$;
(iii) If we denote

$$A(y) = \int_0^y a(u)du, \; y \in R, \tag{10}$$

then

$$\lim |A(y)| = +\infty, \quad \text{as} \quad |y| \to \infty;$$

(iv) There exist positive numbers α, β, such that

$$A(y) < 0 \; \text{ for } \; y < -\alpha \; \text{ or } \; 0 < y < \beta,$$

and

$$A(y) > 0 \; \text{ for } \; -\alpha < y < 0 \; \text{ or } \; y > \beta.$$

Then, the equation (9) has a nontrivial periodic solution.

Remark. The conditions i) – iv) do not assure the uniqueness of the periodic solution to (9). Under extra assumptions, among them the symmetry of $a(y)$, i.e., $a(y) = a(-y)$, and the anti-symmetry of $b(y)$, i.e., $b(-y) = -b(y)$, it can be shown that the nontrivial periodic solution of equation (9) is unique.

Returning to the equation (8), which provides the solitary waves for (1), with $f(y, y')$ given by (5), we notice that it is of the form (9), where

$$a(y) = (1 - c^2)^{-1} g(y), \, b(y) = (1 - c^2)^{-1} h(y). \tag{11}$$

We have chosen to deal in detail with the case $|c| < 1$, which means $(1 - c^2)^{-1} > 0$.

It is obvious that conditions i)−iv) of the theorem stated above will be verified by the functions $a(y)$ and $b(y)$ given by (11), if $g(y)$ and $h(y)$ satisfy the following assumptions:

1) $g(y)$ and $h(y)$ are continuously differentiable maps from R into R;
2) $yh(y) \neq 0$ for $y \neq 0$;
3) If we denote

$$G(y) = \int_0^y g(u)du, \, y \in R, \tag{12}$$

then

$$\lim |G(y)| = +\infty \quad \text{as} \quad |y| \to \infty.$$

4) There exist positive numbers α, β, such that

$$G(y) < 0 \; \text{ for } \; y < -\alpha \; \text{ or } \; 0 < y < \beta,$$

and

$$G(y) > 0 \; \text{ for } \; -\alpha < y < 0 \; \text{ or } \; y > \beta.$$

Consequently, on behalf of the auxiliary theorem stated above, we can formulate the following result in regard to the equation

$$u_{tt} = u_{xx} + g(u)u_x + h(u), \tag{13}$$

which is the special case of (1), with f chosen by (5):

Consider the equation (13), with $g(u)$ and $h(u)$ satisfying the conditions 1)−4) stated above.

Then, there exist periodic solitary waves corresponding to (13), of the form $u(t,x) = = y(x \pm ct)$ where the function y is a periodic solution of the nonlinear Liénard's equation (8).

Remark 1. If $y(x)$ is a periodic solution of the equation (8), say of period $\omega > 0$, then the period of the solution $y(x \pm ct)$, with respect to t, is $\omega/|c|$. In the case $|c| < 1$, all the periods are larger than ω, and there are solutions of arbitrary large period. Estimates for the period ω are difficult to obtain in the general case.

Remark 2. Besides the periodic solution, equation (8) admits also asymptotically periodic solutions that generates asymptotically periodic waves. It is also possible to get solution for (8) which tend asymptotically to the origin $(0,0)$ of the phase plane when $t \to +\infty$ or $t \to -\infty$. All these situations are elementary consequences of the properties of the limit cycles for plane autonomous systems. See, for instance, [5-7].

3. Another Special Case of Equation (1). We shall consider now the case when $f(y, y')$ is independent of y', which means that the wave equation (1) reduces to

$$u_{tt} = u_{xx} + f(u), \tag{14}$$

while the ordinary differential equation for the solitary waves becomes

$$(1 - c^2)y'' + f(y) = 0. \tag{15}$$

This is an equation of the pendulum type and it possesses periodic solutions under adequate conditions on $f(y)$.

Let us assume that $|c| < 1$ again, the case $|c| > 1$ leading to similar assumptions for $f(y)$. This time we will have a family of periodic solutions, depending on a single real parameter. More precisely, in the phase plane, the origin will possess a neighborhood which is filled up with periodic solutions.

Let us take again, as reference, a result from [7, Theorem 6.1]. Formulating directly our hypotheses on $f(y)$, the following result can be stated in regard to the equation (14):

Theorem 2. Consider the equation (14) in which $f(u)$ is continuously differentiable from R into R, such that

$$uf(u) > 0 \quad \text{for} \quad u \neq 0, \qquad f(u) = -f(-u). \tag{16}$$

Then, there exists a family of periodic solitary waves solutions to (14), depending of a real parameter r, $r < r_0$, with $r_0 > 0$ sufficiently small.

Each periodic solitary wave is of the form $y = y(x \pm ct)$, with $y(x)$ a periodic solution of the equation (14).

The proof of Theorem 2 is the immediate consequence of applying to the equation (15) the result mentioned above from [7].

Remark. In the phase space $(y,\ y')$, the trajectories of periodic solution to (15) are symmetric with respect to the origin. The real parameter r can be taken equal to the abscissa of the intersection of the trajectory with the y-axis.

In concluding this paper, we notice the fact that a simple hyperbolic structure described by equation (1) can generate, under adequate assumptions, a variety of solitary waves, including the periodic ones.

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