Entropy of the Shift on $II_1$-representations of the Group $S(\infty)$

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(Presented by Yu. M. Beresanskii)

Abstract. We have obtained the explicit formulae for the CNT-entropy of the shift on $II_1$-representations of the infinite symmetric group $S(\infty)$ in terms of Thoma-parameters.

2000 MSC. 37A35; 37B40; 20C32.

Key words and phrases. CNT-entropy, factor representation, infinite symmetric group.

1. Introduction

Entropy is one of the most important notion in the information theory and the ergodic theory. Initially entropy has appeared in the Claude Shannon’s applied works. Next Kolmogorov and Sinai developed the important invariant, namely the entropy for an automorphism of an Abelian $W^*$-algebra (see [9], [15], [16]). In 1975 the entropy for an automorphism of a non-abelian $W^*$-algebra with a central state was defined by A. Connes and E. Størmer (see [5]). The final definition was given in the paper of Connes, Narnhofer and Tirring in 1987 (see [4]). This one is usually called by the quantum dynamical entropy or the CNT-entropy.

The CNT-entropy is calculated for many non-commutative dynamical systems of the topological, algebraic or physical origin. We consider in our work the dynamical systems generated by the shift automorphism on the $II_1$-representations of the infinite symmetric group $S(\infty)$. The group $S(\infty)$ has been often quoted as a typical example of ICC-groups and hence of groups of non-type $I$. For that reason $S(\infty)$ involves a number of interesting features which not observed in groups of type $I$. Dynamical systems generated by the non-commutative shift have been

Received 25.02.2004
Supported in part by CRDF grant UM1-2546

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investigated beginning from the introduction of the notion of the CNT-entropy. Connes and Størmer obtained the explicit formulae for the non-commutative Bernoulli shift (see [5]). In the work of Størmer and Golodets the similar results was obtained for the binary shift on a CAR-algebra (see [7]). The main examples for which the C*-algebra entropy have been computed, are those of quasifree states of the CAR and CCR-algebras and invariant Bogoliubov (or quasifree) automorphisms (see [2], [6], [11], [12], [14], [17]). In our work [3] the Bogoliubov automorphisms on the $\mathcal{II}_1$-representations of $U(\infty)$ are defined and the explicit formulae for the CNT-entropy are obtained in the case of elementary characters. Using the results of the present work for a low estimation of the CNT-entropy of the shift on the $\mathcal{II}_1$-representations of $U(\infty)$ we obtain the formulae for the Bogoliubov automorphism in the case of a general character (this results will be published in the separate paper).

Denote by $S(2n+1)=S(B_n)$ the group of permutations of the set $B_n = \{-n, \ldots, 0, \ldots, n\}$. If $A$ and $B$ are two sets and $B \subset A$, then we identify $S(B)$ with the subgroup \{\(g \in S(A) : ga = a \forall a \in A\setminus B\)\} of $S(A)$. Let $S(\infty) = \bigcup S(2n+1)$. Thoma has obtained the full description of $\mathcal{II}_1$-factor-representations of group $S(\infty)$. Corresponding normalized characters $\chi^{(S)}_{\alpha,\beta}$ are labelled by a pair of sequences of real numbers \(\{\alpha_i\} = \alpha, \{\beta_i\} = \beta, i = 1, 2, \ldots\), such that $\alpha_i \geq \alpha_{i+1} \geq 0$, $\beta_j \geq \beta_{j+1} \geq 0 \forall i, j \in \mathbb{N}$, $\sum \alpha_i + \sum \beta_j \leq 1$. The value of a character $\chi^{(S)}_{\alpha,\beta}$ on a permutation with a single cycle of length $k$ is equal to

$$\sum_j \alpha_j^k + (-1)^{k-1} \sum_j \beta_j^k$$

(1.1)

Its value on a permutation with several disjoint cycles equals to the product of its values on each cycle. As usual, it is assumed that an empty product equals to 1. In particular, the character of the regular representation of the group $S(\infty)$ corresponds to the sequences $\alpha_j \equiv 0$, $\beta_j \equiv 0$.

The bijection $i \in \mathbb{Z} \rightarrow i + 1 \in \mathbb{Z}$ defines naturally an automorphism $\vartheta_S$ of the group $S(\infty)$, which extends up to the automorphism $\vartheta_S^\chi$ of the $\mathcal{II}_1$-factor built by the representation that corresponds to the character $\chi$. We denote by $H_\chi(\theta)$ the CNT-entropy of an automorphism $\theta$ of the $\mathcal{II}_1$-factor.

The main result of our work is following

**Theorem 1.1.** Let $\chi = \chi^{(S)}_{\alpha,\beta}$, let $\eta(t) = -t \ln t$ and $\gamma = 1 - (\sum \alpha_i + \sum \beta_j)$.

(i) If $\gamma > 0$ then $H_\chi(\vartheta_S^\chi) = \infty$.

(ii) If $\gamma = 0$ then $H_\chi(\vartheta_S^\chi) = \sum_j \eta(\alpha_j) + \sum_j \eta(\beta_j)$. 

2. The Case $\gamma > 0$

In this section we will consider the case $\gamma > 0$.

**Theorem 2.1.** Let $\chi = \chi_{\alpha,\beta}^{(S)}$ and $\gamma = 1 - (\sum \alpha_i + \sum \beta_j) > 0$, then $H_\chi \left( \vartheta_\chi^{(S)} \right) = \infty$.

We will prove several subsidiary statements.

Consider the complex type II$_1$ factor-representation $\Pi_\chi$ of the group $S(\infty)$ which corresponds to the normalize character $\chi$ (see (1.1)). We assume that $\Pi_\chi$ is realized in Hilbert space $H_\chi$ which is the closure of the linear span of vectors $u \in S(\infty)$ with the scalar product $\langle u, v \rangle_\chi = \chi(uv^*)$.

In $H_\chi$ we define the unitary representations $l_\chi$ and $r_\chi$ of the group $S(\infty)$:

$$l_\chi(u)v = uv, \quad r_\chi(u)v = vu^*.$$ \hfill (2.1)

Let us denote by $L_\chi$ ($R_\chi$) the $W^*$-algebra generated by $l_\chi(S(\infty))$ ($r_\chi(S(\infty))$) and denote by $H_\chi(N_1, N_2, \ldots, N_k)$ a CNT-entropy of a system of finite-dimensional subalgebras $N_1, N_2, \ldots, N_k \subset L_\chi$ (see [5]).

If $\mathfrak{A}$ is an operator family and $\mathfrak{A}'$ is the commutant of $\mathfrak{A}$ then $L_\chi' = R_\chi$.

**Definition 2.1.** A normalize character $\chi$ on $G$ is called an indecomposable one if algebra $L_\chi$ ($R_\chi$) is a factor.

**Lemma 2.1.** Let $A \subset B_n$ and let $W^*$-algebra $L_\chi(S(A))$ be generated by operators $l_\chi(g)$ ($g \in S(A)$). If $\chi$ is an indecomposable normalize character on $S(\infty)$ then

$$H_\chi \left( \vartheta_\chi^{(S)} \right) \geq \frac{H_\chi(L_\chi(S(A)))}{2n + 1}$$ \hfill (2.2)

**Proof.** Let $tr_\chi$ be a trace on $L_\chi$ that corresponds to character $\chi$. If $\alpha = (\vartheta_\chi^{(S)})^{2n+1}$, $N_k = \alpha^k(L_\chi(S(A)))$, then the following properties hold true:

i) $N_k$ are pairwise commute for any $k \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers;

ii) if $n_1, n_2 \in \mathbb{Z}$ and $n_1 < n_2$, then $\exists$ a masa $^1$ $\mathfrak{A} \subset \bigvee_{n_1}^{n_2} N_k$ for which $A_k = \mathfrak{A} \cap N_k$ is the masa in $N_k$;

iii) $A = \bigvee_{n_1}^{n_2} A_k$ and $tr_\chi\left( \prod_{k=n_1}^{n_2} a_k \right) = \prod_{k=n_1}^{n_2} tr_\chi(a_k) \forall a_k \in A_k$.

$^1$ maximal abelian subalgebra
From these statements and properties (D), (E) [5] it follows that
\[ H_\chi (N_{n_1}, N_{n_1+1}, \ldots, N_{n_2}) = H_\chi \left( \bigvee_{n_1}^{n_2} A_k \right) = (n_2 - n_1 + 1) H_\chi (\mathcal{L}_\chi (S(\mathcal{A}))). \]

Thus \((2n + 1) H_\chi (\vartheta_S^\lambda) = H_\chi (\alpha) \geq H_\chi (\mathcal{L}_\chi (S(\mathcal{A}))). \)

Next statement allows a lower boundary for the entropy \(H_\chi (\mathcal{L}_\chi (S(\mathcal{A})))\) in a case of the regular representation.

**Lemma 2.2.** Let \(\mathcal{A}\) be the same one as in Lemma 2.1. If \(\chi\) is a character of the regular representation, then there exists a number \(C\) which does not depend on \(\mathcal{A}\) and \(H_\chi (\mathcal{L}_\chi (S(\mathcal{A}))) \geq C \cdot |\mathcal{A}| \cdot \ln |\mathcal{A}|.\)

**Proof.** Let \(|\mathcal{A}| = m\), and let \(\chi^{(\lambda)}\) be a character of an irreducible representation \(\pi_\mathcal{A}\) of the group \(S(m) = S(\mathcal{A})\) which corresponds to the Young diagram \(\lambda\), \(\dim \lambda = \dim \pi_\mathcal{A}\), \(\chi^{(\lambda)}_{\text{norm}} = \frac{\chi^{(\lambda)}}{\dim \lambda}.\) If \(\chi_m\) is a restriction of \(\chi\) on \(S(m)\) and \(|\lambda|\) is the number of boxes in \(\lambda\), then
\[ \chi_m = \sum_{\lambda: |\lambda| = m} \frac{(\dim \lambda)^2}{m!} \chi^{(\lambda)}_{\text{norm}}. \] (2.3)

We denote by \(e_\lambda\) the minimal projection in \(W^*\)-algebra \((\pi_\lambda (S(m)))''\) which is generated by operators \(\pi_\lambda (g)\) \((g \in S(m))\). \(h(p, q)\) will denote the corresponding hook length for a box \((p, q)\) \(\in \lambda\). Recall the well-known hooks-formula
\[ \dim \lambda = m! \cdot \prod_{(p, q) \in \lambda} \frac{1}{h(p, q)}. \] (2.4)

Using (2.3) and (2.4), we obtain
\[ \chi_m (e_\lambda) = \prod_{(p, q) \in \lambda} \frac{1}{h(p, q)}. \] (2.5)

It implies that
\[ H_\chi (\mathcal{L}_\chi (S(\mathcal{A}))) = \sum_{\lambda: |\lambda| = m} - \dim \lambda \cdot \chi_m (e_\lambda) \cdot \ln (\chi_m (e_\lambda)) \]
\[ = \sum_{\lambda: |\lambda| = m} \frac{(\dim \lambda)^2}{m!} \cdot \ln \left( \prod_{(p, q) \in \lambda} h(p, q) \right). \] (2.6)
Using the following inequality belonged to Vershik and Kerov (see [18]) and (2.3)
\[
\exp \left( \frac{c_0}{2} \sqrt{m} \right) \cdot \sqrt{m!} \leq \min_{\lambda: |\lambda|=m} \prod_{(p,q) \in \lambda} h(p,q) \leq \exp \left( \frac{c_1}{2} \sqrt{m} \right) \cdot \sqrt{m!},
\]
where \(c_0\) and \(c_1\) are positive integers, from (2.6) we obtain
\[
H_\chi (\mathcal{L}_\chi (S(A))) \geq \frac{c_0}{2} \sqrt{m} + \frac{1}{2} \cdot \ln (m!).
\]
So the statement of our lemma follows from Stirling’s formula. \(\square\)

Now let us take for \(\chi\) an arbitrary indecomposable normalize character on \(S(\infty)\). If \(\mathcal{M}\) is an injective finite factor with normalize trace \(tr\), then there is a representation
\[
\pi_\chi : S(\infty) \rightarrow U(\mathcal{M})
\]
with the property
\[
\chi(g) = tr (\pi_\chi (g)).
\]
Here \(U(\mathcal{M})\) denotes a group of unitary operators in \(\mathcal{M}\).

Consider the following operator limit in the weak operator topology
\[
\lim_{n \rightarrow \infty} \pi_\chi ((i, n)) = A_i, \tag{2.7}
\]
where \((i, n) \in S(\infty)\) is a transposition. It is obviously, that \(A_i = A_i^*\). Let \(\mu\) be a spectral measure of operator \(A_i\):
\[
\int x^k \mu(dt) = tr \left( A_i^k \right) \forall \ k \in \mathbb{N}.
\]
We denote by \(\mathbb{N}/g\) a set of orbits of a permutation \(g\) on the set \(\mathbb{N}\). Denote by \(|p|\) the cardinality of an orbit \(p \in \mathbb{N}/g\). The following statement belongs to A. Okounkov (see [13]).

**Lemma 2.3.** The following properties are true:

a) \(A_i A_j = A_j A_i \ \forall \ i, j \in \mathbb{Z}\) and \(tr \left( \prod_l A_{j_l}^{k_l} \right) = \prod_l tr \left( A_{j_l}^{k_l} \right) \ \forall \ k_l \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} ;
\)
b) \(\pi_\chi (g) A_i \pi_\chi (g^{-1}) = A_{g(i)} ;
\)
c) \(\text{supp} \mu \subset [-1, 1] , \ \text{the measure} \ \mu \ \text{is discrete and} \ \forall \ \varepsilon > 0 \ \text{a set} \ \{}[-1, -\varepsilon] \cup [\varepsilon, 1]\ \text{contains at the most} \ \frac{2}{\varepsilon} \ \text{its atoms} ;\)
d) Let \( f_i, g_i \) (\( i \in \mathbb{Z} \)) are functions on \([-1, 1]\) which are pointwise limits of uniformly bounded sequences of continuous functions. If all of \( f_i, g_i \) (\( i \in \mathbb{Z} \)) but finitely many identically equal to 1, then

\[
\text{tr} \left( \prod_{i \in \mathbb{Z}} \bar{g}_i(A_i) \pi_X(g) \prod_{i \in \mathbb{Z}} f_i(A_i) \right) = \prod_{p \in \mathbb{N}/g} \int x^{|p|-1} \prod_{i \in p} f_i(t) g_i(t) d\mu;
\]

\( e) \) \( \forall x \neq 0 \) \( \nu(x) = \frac{\mu(x)}{|x|} \in \mathbb{Z}_+; \)

\( f) \) if \( \chi = \chi^{(s)}_{\alpha, \beta} \) (see (1.1)), \( x \neq 0 \) and \( x \in \text{supp} \mu \), then \( \exists i \in \mathbb{N} \), for which

\[
\begin{align*}
\alpha_i &= x, \ldots, \alpha_i + \nu(x) - 1 = x & \text{if } x > 0, \\
\beta_i &= |x|, \ldots, \beta_i + \nu(x) - 1 = |x| & \text{if } x < 0.
\end{align*}
\]

Denote by \( \delta_x \) the function that equals to 1 at the point \( x \), and that equals to 0 at all the rest points. Let \( E_i = \delta_0(A_i) \).

The next statement easily follows from the previous lemma.

**Corollary 2.1.** Let \( \chi = \chi^{(s)}_{\alpha, \beta}, \gamma = \text{tr} (E_n) = 1 - \sum_i (\alpha_i + \beta_i) \), and let \( A_k = \{i_1, i_2, \ldots, i_k\} \) be a set of different numbers from \( \mathbb{Z} \). If \( E_{A_k} = \prod_{j=1}^k E_{i_j}, \gamma > 0 \), then for \( g \in S(A) \)

\[
\varphi_{\gamma, k}(g) = \gamma^{-k} \cdot \text{tr} (E_{A_k} \pi_X(g) E_{A_k}) = \begin{cases} 
1 & \text{if } g = e, \\
0 & \text{otherwise}.
\end{cases}
\]

From here and from lemma 2.2 it follows the next

**Lemma 2.4.** If \( \gamma \in [0, 1[ \), and if \( E_i \) (\( |i| \leq n \)) and \( \pi_X(S(B_n)) \) generates a \( W^* \)-algebra \( \mathcal{M}_n \), then there is some constant \( C_1 \) which doesn’t depends on \( n \) and such that

\[
H_{\chi}(\mathcal{M}_n) \geq C_1 \cdot n \ln n.
\]

**Proof.** Let us use the notations of Corollary 2.1. By Lemma 2.2 there exists a constant \( C \) which does not depend on \( k \) and \( C \) is such that

\[
H_{\varphi_{\gamma, k}}(E_{A_k} \pi_X(S(A_k)) E_{A_k}) \geq C \cdot k \ln k.
\]
Taking into consideration this result, we obtain

\[ H_\chi(\mathcal{M}_n) \geq \sum_{k=0}^{n} \sum_{\lambda_k:|\lambda_k|=k} \binom{n}{k} (1-\gamma)^{n-k} \gamma^k \frac{(\dim \lambda_k)^2}{k!} \]
\[ \times \left[ \ln \left( \prod_{(p,q)\in\lambda_k} h(p,q) \right) - k \ln \gamma - (n-k) \ln(1-\gamma) \right] \]
\[ \geq -n (\gamma \ln \gamma + (1-\gamma) \ln(1-\gamma)) + \mathcal{C} \sum_{k=0}^{n} \binom{n}{k} (1-\gamma)^{n-k} \gamma^k k \ln k. \]

(2.8)

Now we take a constant \( d > 0 \) for which

\[ \sum_{k=[n\gamma-d\sqrt{n}]}^{[n\gamma+d\sqrt{n}]} \binom{n}{k} (1-\gamma)^{n-k} \gamma^k > \frac{1}{2} \quad \forall \ n \in \mathbb{N}. \]

Taking into account this and (2.8), we have

\[ H_\chi(\mathcal{M}_n) \geq -n (\gamma \ln \gamma + (1-\gamma) \ln(1-\gamma)) + \mathcal{C} \left[ n\gamma - d\sqrt{n} \right] \ln \left[ n\gamma - d\sqrt{n} \right]. \]

Thus, the statement of Lemma 2.4 is proved. \( \square \)

Proof of Theorem 2.1. If \( \gamma = 1 \), then the statement of Theorem 2.1 follows from Lemmas 2.1 and 2.2. Let \( \gamma < 1 \). Using a method we have proved Lemma 2.1, we receive the following estimation

\[ H_\chi(\mathcal{M}_n) \geq H_\chi(\mathcal{M}_n) \]
\[ \geq H_\chi(\mathcal{M}_n) \]
\[ \geq \frac{H_\chi(\mathcal{M}_n)}{2n+1} \] (see Lemma 2.4).

Thus, the statement of Theorem 2.1 follows from Lemma 2.4. \( \square \)

3. The Case of \( \gamma = 0 \)

In this section we will present two different entropy estimation methods developed for the case of finite cardinality of set \( I = \{ i : \alpha_i > 0 \} \cup \{ i : \beta_i > 0 \} \) and for the case of infinite one correspondingly. First method is based on the important formulaes from the theory of symmetric functions. The second one uses the structural properties of von Neumann factors constructed by the representations of \( S(\infty) \). It will be clear, that the case of finite cardinality can be included in the second one, but we would like to show special technic in the Subsection 3.1.
### 3.1. The Subcase $|I| < \infty$

In this Section we will prove the following theorem.

**Theorem 3.1.** Let $\eta(t) = -t \ln t$, $\sum \alpha_i + \sum \beta_j = 1$, $\chi = \chi^{(S)}_{\alpha,\beta}$ and let $\mathcal{N} \in \mathbb{N}$ exist for which $\alpha_j = \beta_j = 0 \forall j > \mathcal{N}$. Then

$$H_{\chi} (\vartheta^{S}_S) = \sum_j \eta(\alpha_j) + \sum_j \eta(\beta_j).$$

Consider the restriction of $\chi$ onto a finite symmetric group $S(A)$. The characters of the finite symmetric group $S(A)$ are labeled by the Young diagrams with $|A|$ boxes. Let $\chi^{(\lambda)}$ be a (non normalized) character corresponding to an irreducible representation $\lambda$. The restriction $\chi|_{S(A)}$ to the group $S(A)$ is a non-negative linear combination of the functions $\chi^{(\lambda)}$

$$\chi|_{S(A)} = \sum_{\lambda:|\lambda|=|A|} \tilde{s}_\lambda(\alpha,\beta) \cdot \chi^{(\lambda)}. \tag{3.1}$$

The Fourier coefficient $\tilde{s}_\lambda(\alpha,\beta)$ is given by the extended Schur function (see [8]), which can be formally defined by Jacoby-Trudi determinant

$$\tilde{s}_\lambda(\alpha,\beta) = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \ldots & h_{\lambda_1+m-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \ldots & h_{\lambda_2+m-2} \\ h_{\lambda_3-2} & h_{\lambda_3-1} & h_{\lambda_3} & \ldots & h_{\lambda_3+m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_{m-m+1}} & h_{\lambda_{m-m+2}} & h_{\lambda_{m-m+3}} & \ldots & h_{\lambda_m} \end{vmatrix}, \tag{3.2}$$

where the extended complete homogeneous symmetric functions $h_l = h_l(\alpha,\beta)$ arise as the coefficients of the generating series

$$e^{z\gamma} \prod_{j=1}^{\infty} \frac{1 + z\beta_j}{1 - z\alpha_j} = 1 + \sum_{l=1}^{\infty} h_l(\alpha,\beta)z^l.$$

We denote by $d = d(\lambda)$ the number of diagonal boxes in the Young diagram $\lambda$ and we will use the Frobenius notation [10]

$$\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d).$$

Here $p_i = \lambda_i - i$ is a number of boxes in the $i-$th row of $\lambda$ on the right of the $i-$th diagonal box; likewise, $q_i = \lambda'_i - i$ is the number of boxes in the $i-$th column of $\lambda$ below the $i-$th diagonal box ($\lambda'$ stands for the transposed diagram).
Lemma 3.1. Let $\alpha = \{\alpha_i\}_{i=1}^{\infty}, \beta = \{\beta_i\}_{i=1}^{\infty}$ be Thoma-parameters, $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) = 1, N_\alpha = \max \{i \in \mathbb{N} : \alpha_k > 0\}, N_\beta = \max \{i \in \mathbb{N} : \beta_k > 0\}$. If $\max \{N_\alpha, N_\beta\} < \infty$, then $s_\lambda(\alpha, \beta) = 0$ in each of the following cases

1. $d(\lambda) > d = \max\{N_\alpha, N_\beta\}$;
2. $\lambda_i > d \forall i = N_\beta + 1, \ldots, d$;
3. $\lambda_i' > d \forall i = N_\alpha + 1, \ldots, d$.

Proof. We consider a sequence of the Young diagrams

$$\chi^{(2n+1)} = \left(\frac{p_1^{(2n+1)}}{2n+1}, \ldots, \frac{p_d^{(2n+1)}}{2n+1} | \frac{q_1^{(2n+1)}}{2n+1}, \ldots, \frac{q_d^{(2n+1)}}{2n+1}\right)$$

with properties:

1. $|\chi^{(2n+1)}| = 2n + 1$ and $d = d(\chi^{(2n+1)}) = \max\{N_\alpha, N_\beta\}$ for $n$ sufficiently great;
2. $\alpha_i = \lim_{n \to \infty} \frac{p_i^{(2n+1)}}{2n+1}, \beta_i = \lim_{n \to \infty} \frac{q_i^{(2n+1)}}{2n+1} \forall i = 1, \ldots, d$.

It follows from the approximation Theorem [19] that

$$\chi(g) = \chi^{(S)}_{\alpha, \beta}(g) = \lim_{n \to \infty} \frac{\chi^{\chi^{(2n+1)}}_{\chi^{(2n+1)}}(g)}{\dim \chi^{(2n+1)}} \forall g \in S(\infty).$$

Using this claim, property 1) and the Young branching rule

$$\chi^{(\Lambda)} |_{S(|\Lambda|)} = \sum_{\lambda : \Lambda \setminus \lambda} \chi^{(\lambda)},$$

where the notation $\Lambda \setminus \lambda$ means that diagram $\lambda \subset \Lambda$ is obtained from the diagram $\Lambda$ by removing a box, we obtain the statement of the lemma. \hfill \Box

Further we will need the Berele-Regev formula (see [1]) for the supersymmetric Schur functions $s_\lambda$

$$s_\lambda(x_1, \ldots, x_d; y_1, \ldots, y_d) = \frac{\det [x_i^{p_j}]_{i,j=1}^{d}}{V(x_1, \ldots, x_d)} \cdot \frac{\det [y_i^{q_j}]_{i,j=1}^{d}}{V(y_1, \ldots, y_d)} \prod_{i,j=1}^{d} (x_i + y_j).$$

(3.3)

Here $\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d), V(\ldots)$ is the Vandermonde determinant and the parameters $x_1, \ldots, x_d$, as well as $y_1, \ldots, y_d$, are assumed to be pairwise distinct.
If \( \sum_{i=1}^{\infty} (\alpha_i + \beta_i) = 1 \), then the extended Schur (3.1) function coincides with the supersymmetric Schur function
\[
\tilde{s}_\lambda(\alpha, \beta) = s_\lambda(\alpha, \beta).
\]

Now we will obtain the lower boundary for entropy \( H_\chi(\mathcal{L}_\chi(S(n))) \) (see Lemma 2.1).

**Lemma 3.2.** Let parameters \( \alpha = \{\alpha_i\}_{i=1}^{\infty} \) and \( \beta = \{\beta_i\}_{i=1}^{\infty} \) satisfy the conditions of Lemma 3.1, \( \chi = \chi_{\alpha, \beta}^{(S)} \). Then \( \forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N} \) for which
\[
H_\chi(\mathcal{L}_\chi(S(n))) \geq -n(1 - \varepsilon) \left\{ \sum_{j=1}^{N_\alpha} [\alpha_j - \varepsilon] \cdot \ln \alpha_j + \sum_{j=1}^{N_\beta} [\beta_j - \varepsilon] \cdot \ln \beta_j \right\} + \mathfrak{N} \ln n \quad \forall n > N(\varepsilon),
\]

where \( \mathfrak{N} \) is a constant, which does not depend on \( n \).

**Proof.** Let \( Y_n(d) \) be a set of Young diagrams \( \lambda \) such that \( |\lambda| = n \) and \( d(\lambda) \leq d \). For \( k < d \) we define two sets
\[
Y_n(d, k) = \{ \lambda \in Y_n(d) : \lambda_i' \leq d \forall i = k + 1, k + 2, \ldots \},
\]
\[
Y_n'(d, k) = \{ \lambda \in Y_n(d) : \lambda_i \leq d \forall i = k + 1, k + 2, \ldots \}.
\]

We assume, that \( N_\alpha \geq N_\beta \). By Lemma 3.1, we have
\[
\chi|_{S(n)} = \sum_{\lambda \in Y_n(N_\alpha, N_\beta)} s_\lambda(\alpha, \beta) \cdot \chi^{(\lambda)}. \quad (3.4)
\]

Let
\[
Y_n(d, k, \varepsilon) = \{ \lambda \in Y_n(d, k) : \lambda_i' = N_\alpha \quad \forall i = N_\beta + 1, \ldots, N_\alpha, \quad |p_i(\lambda) - n\alpha_i| < n\varepsilon \text{ and } |q_j(\lambda) - n\beta_j| < n\varepsilon \quad \forall i = 1, \ldots, d; \quad j = 1, \ldots, k \}. \quad (3.5)
\]

Using (3.4) and (3.5), by the approximation Theorem [19] we obtain, that there exists \( N(\varepsilon) \in \mathbb{N} \) for which
\[
1 \geq \sum_{\lambda \in Y_n(N_\alpha, N_\beta, \varepsilon)} \dim \lambda \cdot s_\lambda(\alpha, \beta) > 1 - \varepsilon \quad \forall n > N(\varepsilon). \quad (3.6)
\]
Formula (3.3) can be extended by a continuity to the case, when the number of parameters \( x_1, \ldots, x_n \) is not equal to the number of parameters \( y_1, \ldots, y_m \). We assume that, the parameters \( \{\alpha_1 \geq \ldots \geq \alpha_{N\alpha} > 0\} \) are pairwise distinct as well as the parameters \( \{\beta_1 \geq \ldots \geq \beta_{N\beta} > 0\} \). The next statement is obtained for the diagram \( \lambda = (p_1, \ldots, p_{N\alpha}|q_1, \ldots, q_{N\alpha}) \in Y_n(N_{\alpha}, N_{\beta}, \varepsilon) \) from relation (3.3) by passing to the limit \( (\beta_{N\beta+1} \to 0, \ldots, \beta_{N\beta} \to 0) \)

\[
\begin{align*}
s_{\lambda}(\alpha_1, \ldots, \alpha_{N\alpha}; \beta_1, \ldots, \beta_{N\beta}) &= \frac{\det[\alpha_{ij}]_{i,j=1}^{N\alpha}}{V(\alpha_1, \ldots, \alpha_{N\alpha})} \\
&\times \frac{\det[\beta_{ij}]_{i,j=1}^{N\beta}}{V(\beta_1, \ldots, \beta_{N\beta})} \prod_{i=1}^{N\alpha} \frac{1}{\alpha_i - N\alpha} \prod_{j=1}^{N\beta} (\alpha_i + \beta_j).
\end{align*}
\]

(3.7)

Now we consider the case, when there are the coincident parameters. Let \( \{n_i(\alpha)\}_{i=1}^{k_{\alpha}} \) and \( \{n_i(\beta)\}_{i=1}^{k_{\beta}} \) be subsets in \( \mathbb{N} \) with the properties:

\[
\begin{align*}
\sum_{i=1}^{k_{\alpha}} n_i(\alpha) &= N_{\alpha}, & \sum_{i=1}^{k_{\beta}} n_i(\beta) &= N_{\beta}, \\
\alpha_{n_1(\alpha)+\ldots+n_j(\alpha)+1} = \ldots = \alpha_{n_1(\alpha)+\ldots+n_j(\alpha)+n_{j+1}(\alpha)} &= t_j, \\
\beta_{n_1(\beta)+\ldots+n_j(\beta)+1} = \ldots = \beta_{n_1(\beta)+\ldots+n_j(\beta)+n_{j+1}(\beta)} &= s_j,
\end{align*}
\]

(3.8)

the parameters \( t_1, \ldots, t_{k_{\alpha}}, \) are pairwise distinct as well as \( s_1, \ldots, s_{k_{\beta}}. \)

If

\[
T_{jk} = \begin{cases} 
  t_r^{p_k} & \text{if } j = \sum_{i=1}^{r+1} n_i(\alpha), \\
  \prod_{i=1}^{m-1} (p_k - i + 1) t_r^{p_k-m+1} & \text{if } j = -m + \sum_{i=1}^{r+1} n_i(\alpha), 
\end{cases}
\]

where \( m = 1, \ldots, n_{r+1}(\alpha) - 1; \)

\[
S_{jk} = \begin{cases} 
  s_r^{q_k} & \text{if } j = 1 + \sum_{i=1}^{r+1} n_i(\beta), \\
  \prod_{i=1}^{m-1} (q_k - i + 1) s_r^{q_k-m+1} & \text{if } j = -m + \sum_{i=1}^{r+1} n_i(\beta), 
\end{cases}
\]

where \( m = 1, \ldots, n_{r+1}(\beta) - 1; \)

then we can rewrite (3.7) as follows:

\[
s_{\lambda}(\alpha, \beta) = \frac{\det T}{\prod_{1 \leq l < j \leq k_{\alpha}} (t_l - t_j)^{n_l(\alpha) \cdot n_j(\alpha) \cdot (n_j(\alpha) - 1)! \cdot (n_l(\alpha) - 1)!}}.
\]
\[
\times \prod_{1 \leq l < j \leq k_{\beta}} (s_l - s_j)^{n_i(\beta) - n_j(\beta)} \cdot (n_j(\beta) - 1)! (n_l(\beta) - 1)!
\]

\[
\times \prod_{j=1}^{k_{\alpha}} \prod_{i=1}^{k_{\beta}} (t_j + s_i)^{n_j(\alpha) - n_i(\beta)} .
\] (3.9)

Using inequality

\[
x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \geq x_1^{n_1(1)} x_2^{n_2(2)} \cdots x_k^{n_k(\pi(k))},
\]

where \(\pi\) is a permutation, \(0 < x_k \leq \ldots \leq x_2 \leq x_1\), \(n_i \in \mathbb{N}\) (\(1 \leq i \leq k\)) and \(0 < n_k \leq \ldots \leq n_2 \leq n_1\), from (3.9) we have

\[
P_T(p_1, \ldots, p_{N_{\alpha}}) \prod_{i=1}^{k_{\alpha}} t_i^{n_i(\alpha)} \left(\frac{2p_i - n_i(\alpha) + 1}{2}\right) \prod_{i=1}^{N_{\alpha}} \left[\frac{N_{\alpha} - N_{\beta}}{\alpha_i}\right]
\]

\[
P_S(q_1, \ldots, q_{N_{\beta}}) \prod_{i=1}^{k_{\beta}} s_i^{n_i(\beta)} \left(\frac{2q_i - n_i(\beta) + 1}{2}\right) \prod_{i=1}^{N_{\beta}} \left[\frac{N_{\alpha} - N_{\beta}}{\beta_i}\right]
\]

\[
\times \prod_{1 \leq l < j \leq k_{\beta}} (s_l - s_j)^{n_i(\beta) - n_j(\beta)} \cdot \prod_{j=1}^{k_{\alpha}} \prod_{i=1}^{k_{\beta}} (t_j + s_i)^{n_j(\alpha) - n_i(\beta)} .
\] (3.10)

Here \(P_T(P_S)\) is a polynomial of \(N_{\alpha} - k_{\alpha} (N_{\beta} - k_{\beta})\) degree with coefficients which do not depend on \(n\). Thus, we have

\[
H_{\chi} \left(\mathcal{L}_\chi \left( S \left( n \right) \right) \right) = - \sum_{\lambda: |\lambda| = n} \dim \lambda \cdot s_{\lambda}(\alpha, \beta) \cdot \ln s_{\lambda}(\alpha, \beta)
\]

\[
\geq - \sum_{\lambda \in Y_n(N_{\alpha}, N_{\beta})} \dim \lambda \cdot s_{\lambda}(\alpha, \beta) \cdot \ln s_{\lambda}(\alpha, \beta)
\]

see (3.10), (3.8)

\[
\geq \sum_{\lambda \in Y_n(N_{\alpha}, N_{\beta})} \dim \lambda \cdot s_{\lambda}(\alpha, \beta)
\]

\[
\times \left[ \left( \sum_{j=1}^{N_{\alpha}} p_j \cdot \ln \alpha_j + \sum_{j=1}^{N_{\beta}} q_j \cdot \ln \beta_j \right) + (N_{\alpha} + N_{\beta} - k_{\alpha} - k_{\beta}) \ln n + C(\alpha, \beta) \right].
\]

Here \(C(\alpha, \beta)\) is a constant that does not depend on \(n\). From here, taking into account (3.5) and (3.6), we obtain the statement of the lemma. The case when \(N_{\alpha} < N_{\beta}\) can be considered analogous by taking \(Y_n(\cdot, \cdot)\) instead of \(Y_n'(\cdot, \cdot)\). \(\square\)
3.2. The Subcase of Infinite Cardinality

Next we consider the case of the infinite number of nonzero parameters \( \{\alpha_i\} = \alpha, \{\beta_i\} = \beta \) and obtain a lower boundary for the entropy \( H_\chi (M_n) \), where \( M_n \) is generated by \( A_0 \) and \( \pi_\chi(S(n)) \) as a \( W^* \)-algebra.

**Lemma 3.3.** If \( \sum \alpha_i + \sum \beta_j = 1 \), then

\[
H_\chi (\vartheta_\chi S) \geq \sum_i \left( \frac{\eta(\nu(\alpha_i) \cdot \alpha_i)}{\nu(\alpha_i)} + \frac{\eta(\nu(\beta_i) \cdot \beta_i)}{\nu(\beta_i)} \right),
\]

where \( \nu \) is the multiplicity function (see Lemma 2.3).

**Proof.** Let \( \chi = \chi_{\alpha,\beta}^{(S)} \) and let \( \pi_\chi \) be the representation that corresponds to \( \chi \). We denote by \( A \) the \( W^* \)-algebra which is generated by operators \( \{A_i\}_{i \in \mathbb{Z}} \) (see Lemma 2.3). Since \( \vartheta_\chi S (A_i) = A_{i+1} \) (2.7), \( \vartheta_\chi S \) restricts to an automorphism of \( A \). So we get, using properties \( a), c), d), e), f) \) of Lemma 2.3, that the Abelian dynamical system \( (A, \vartheta_\chi S, tr) \) is the classical Bernoulli shift with the entropy

\[
\sum_i \left( \frac{\eta(\nu(\alpha_i) \cdot \alpha_i)}{\nu(\alpha_i)} + \frac{\eta(\nu(\beta_i) \cdot \beta_i)}{\nu(\beta_i)} \right).
\]

\[\square\]

Let us consider the following union \( \{\alpha_i\} = \bigcup_j U_j \), where \( \forall \alpha_k, \alpha_l \in U_j, \alpha_k = \alpha_l, \forall \alpha_k \in U_j, \alpha_l \in U_m, \alpha_k > \alpha_l \) if \( j < m \). Next we define \( \alpha' = \{\alpha'_i\} \) such that \( \forall i \alpha'_i \in U_i, \alpha'_i \neq 0 \) and \( \forall i, j, i \neq j \alpha'_i \neq \alpha'_j \). In the same way we define the sequence \( \beta' = \{\beta'_i\} \).

Let \( a_i = \nu(\alpha'_i) \alpha'_i, b_i = \nu(\beta'_i) \beta'_i \) and let

\[
N_{\alpha,\beta} (m, k_\alpha, k_\beta; D) = \left\{ (m_{1, \alpha}^1, \ldots, m_{k_\alpha}^\alpha, m_{1, \beta}^1, \ldots, m_{k_\beta}^\beta, m_{k_\alpha+k_\beta+1}^\alpha) \in \bigtimes_{j=1}^{k_\alpha+k_\beta+1} \mathbb{N} : \right.
\]

\[
\bigwedge_{i=1}^k \left( a_i m - D\sqrt{m} \leq m_i^\alpha \leq a_i m + D\sqrt{m} \right)
\]

\[
\bigwedge_{j=1}^{k_\alpha} \left( b_j m - D\sqrt{m} \leq m_j^\beta \leq b_j m + D\sqrt{m} \right)
\]

\[
\bigwedge \left( \sum_{j=1}^{k_\alpha} m_j^\alpha + \sum_{j=1}^{k_\beta} m_j^\beta + m_{k_\alpha+k_\beta+1} = m \right)
\]

\[
\forall i = 1, 2, \ldots, k_\alpha; j = 1, 2, \ldots, k_\beta \right\}.
\]

The next statements follows from the central limit theorem.
Lemma 3.4. Let $\sum \alpha_i + \sum \beta_j = 1$, $a_j = \nu(\alpha'_j) \alpha'_j$, $b_j = \nu(\beta'_j) \beta'_j$,

$$\gamma_{kl} = 1 - \sum_{j=1}^{k} a_j - \sum_{j=1}^{l} b_j$$

and let $\delta_1$, $\delta_2$ be given. Then there are $N(\delta_1)$, $N(\delta_1,\delta_2) \in \mathbb{N}$ and $D = D(\delta_1,\delta_2) > 0$ with properties:

i) $\gamma_{kl} < \delta_1 \forall k, l \geq N(\delta_1)$;

ii) if $k_\alpha = \min \{ N(\delta_1), |\alpha'| \}$, $k_\beta = \min \{ N(\delta_1), |\beta'| \}$, $m \geq N(\delta_1,\delta_2)$, then

$$\sum_{\tilde{m} \in N_{\alpha,\beta}(m,k_\alpha,k_\beta,D)} m! \cdot \prod_{j=1}^{k_\alpha} a_j^m \prod_{j=1}^{k_\beta} b_j^m \cdot \gamma_{k_\alpha+k_\beta+1} > 1 - \delta_2,$$

where $\tilde{m} = (m_\alpha^1, \ldots, m_\alpha^{k_\alpha}, m_\beta^1, \ldots, m_\beta^{k_\beta}, m_{k_\alpha+k_\beta+1})$ and $D$, $N(\delta_1,\delta_2)$ are constants which do not depend on $m$.

Let $E_i(x) = \delta_x(A_i)$ (see Lemma 2.3) and let $A_k = \{i_1, i_2, \ldots, i_k\}$ be a set of different numbers from $\mathbb{Z}$. We denote by $E_{A_k}(x)$ projection $\prod_{j=1}^{k} E_{i_j}(x)$. If $g \in S(A_k)$ then by Lemma 2.3 b)

$$[E_{A_k}(x), \pi_\chi(g)] = 0. \quad (3.12)$$

Therefore, the positive definite function $\tau_{A_k}$ on $S(A_k)$, which is defined by formula

$$\tau_{A_k,x}(g) = \frac{tr(E_{A_k}(x)\pi_\chi(g))}{tr(E_{A_k}(x))}, \quad (3.13)$$

where $x \in \{\alpha'\} \cup \{\beta'\}$, is the normalize character.

The next Lemma is an auxiliary one.

Lemma 3.5. The next “dual” formula for extended Schur functions

$$\tilde{s}_\lambda(\alpha, \beta) = \tilde{s}_\lambda'(\beta, \alpha), \quad (3.14)$$

where $\lambda'$ is the transposed diagram for $\lambda$, is valid.

Proof. The formula (3.14) is the generalization of the formula (2.9') from [10]. We will repeat the main ideas of that proof as applied to our case. Let us denote

$$H_{(\alpha,\beta)}(z) = e^{\sum_{j=1}^{\infty} \frac{1 + z\beta_j}{1 - z\alpha_j}} = 1 + \sum_{l=1}^{\infty} h_l(\alpha, \beta)z^l. \quad (3.15)$$
Then

\[ H_{\alpha,\beta}(z)H_{\beta,\alpha}(-z) \equiv 1. \]  \hspace{1cm} (3.16)

Let us consider two matrices

\[ H = \left( h_{i-j}(\alpha, \beta) \right)_{0 \leq i,j \leq N} \]  \hspace{1cm} (3.17)

and

\[ \tilde{H} = \left( (-1)^{i-j} h_{i-j}(\beta, \alpha) \right)_{0 \leq i,j \leq N}, \]  \hspace{1cm} (3.18)

where \( N \) is some positive integer. We remind that \( h_k(\alpha, \beta) = 0 \) for \( k < 0 \) and hence the both matrices are upper-triangular with

\[ \det \tilde{H} = \det H = 1. \]  \hspace{1cm} (3.19)

Moreover, in view of (3.16)

\[ \tilde{H} H = H \tilde{H} = I \]  \hspace{1cm} (3.20)

holds. Hence \( \tilde{H} = H^{-1} \). Let \( H' \) be the transposed matrix for \( H \), \( M \) is an arbitrary minor of the matrix \( H \) and \( A \) is the algebraic adjunct corresponding to the minor \( M' \) of the matrix \( H' \) with the same numbers of columns and rows as the numbers of ones in \( M \). By the Laplace theorem and by the (3.19)-(3.20) we obtain the equation \( M = A \).

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) be a Young diagram, \( \lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m) \) be the transposed diagram. Then by the (3.2) \( s_\lambda(\alpha, \beta) \) can be consider as the minor of the matrix \( H \) with the raw numbers \( \lambda_i - i + n, 1 \leq i \leq n \) and the column numbers \( n - j, 1 \leq j \leq n \). It is well-known that the \( m+n \) numbers \( \lambda_i - i + n, 1 \leq i \leq n \) and \( (m+n-1) - (\lambda'_j - j + m) = n - 1 - \lambda'_j + j, 1 \leq j \leq m \) are the permutation of the \( \{0, 1, 2, \ldots, m + n - 1\} \) (see [10]). Below we assume that the dimension of the matrixes \( N = m+n-1 \). Then the corresponding algebraic adjunct has the raw numbers \( n - 1 - \lambda'_i + i, 1 \leq i \leq m \) and the column numbers \( n-1+j, 1 \leq j \leq m \). Since the elements of the matrix \( H' \) look like \((-1)^{j-i} h_{j-i}(\beta, \alpha)\) the algebraic adjunct consists of such elements \((-1)^{\sum_{i=1}^{n} (\lambda_i - i + n)} - \sum_{j=1}^{m} (n - j) = |\lambda|\). Thus

\[ s_\lambda(\alpha, \beta) = \det \left( h_{\lambda+j-i}(\alpha, \beta) \right)_{1 \leq i,j \leq n} \]
\[ = (-1)^{|\lambda|} \det \left( (-1)^{\lambda'_i + j - i} h_{\lambda'_i + j - i}(\beta, \alpha) \right)_{1 \leq i,j \leq m} \]
\[ = \det \left( h_{\lambda'_i + j - i}(\beta, \alpha) \right)_{1 \leq i,j \leq m} = s_{\lambda'}(\beta, \alpha), \]

\( \square \)
Lemma 3.6. Let \( l(g) \) \((g \in S(A_k))\) is the number of cycles of a permutation \( g \). Then
\[
\tau_{A_k,x}(g) = \frac{(\text{sign } x)^{k-l(g)}}{\nu(x)^{k-l(g)}}. \tag{3.21}
\]
Therefore, \( \tau_{A_k,x} \) is the restriction of characters \( \chi_{\alpha_\nu(x),0}^{(S)} \) for \( x \in \alpha' \)
\((\chi_{0,\beta_\nu(x)}^{(S)} \text{ for } x \in \beta')\) to the group \( S(A_k) \). Here \( \alpha_\nu, \beta_\nu = \{\nu^{-1}, \ldots, \nu^{-1}\} \).

Proof. We denote by \( A_k/g \) a set of orbits of the permutation \( g \) on the set \( A_k \). If \( \mu \) is a spectral measure of operator \( A_k \), then \( \mu(x) = \nu(x) \cdot |x| \) \((\text{Lemma 2.3 e})\) and by (Lemma 2.3 d) we obtain
\[
\tau_{A_k,x}(g) = \prod_{p \in A_k/g} \left[ x^{\nu^{-1}} \mu(x) \right] = \frac{x^{k-l(g)} |x|^{\nu^{-1}} \mu(g)(x)}{|x|^k \cdot \nu^k(x)} = \frac{(\text{sign } x)^{k-l(g)}}{\nu^{k-l(g)}(x)}. \tag{3.22}
\]

Let parameters \( \alpha = \{\alpha_i\}_{i=1}^\infty \) and \( \beta = \{\beta_i\}_{i=1}^\infty \) satisfy the conditions of Lemma 3.2, \( \chi = \chi_{\alpha,\beta}^{(S)} \), the W*-algebra \( \mathcal{L}_\chi(S(A_k)) \) be generated by operators \( \pi_\chi(S(A_k)) \). We denote by \( \mathcal{C}_k^\infty \) the center of the W*-algebra \( \mathfrak{M}_k^\infty(A_k) = E_{A_k}(x) \mathcal{C}_k^\infty(S(A_k)) \).

At first we assume that \( x > 0 \). Then from (3.1) and Lemma 3.4 we obtain
\[
\tau_{A_k,x}(\chi)^{(S)}_{\alpha_\nu(x),0} = \sum_{\lambda:|\lambda|=k} \tilde{s}_\lambda \left( \alpha_\nu(x), 0 \right) \cdot \chi^{(\lambda)}. \tag{3.23}
\]

The coefficients \( \tilde{s}_\lambda \left( \alpha_\nu(x), 0 \right) \) in the expansion can be easily evaluated by using (3.1)
\[
\tilde{s}_\lambda \left( \alpha_\nu(x), 0 \right) = |\nu(x)|^{-k}. \tag{3.24}
\]

If \( x < 0 \), then
\[
\tau_{A_k,x}(\chi)^{(S)}_{0,\beta_\nu(x)} = \sum_{\lambda:|\lambda|=k} \tilde{s}_\lambda \left( 0, \beta_\nu(x) \right) \cdot \chi^{(\lambda)} \text{ and by the Lemma 3.5}
\]
\[
\tilde{s}_\lambda \left( 0, \beta_\nu(x) \right) = \tilde{s}_{\lambda'} \left( \beta_\nu(x), 0 \right) = |\nu(x)|^{-k}. \tag{3.25}
\]
Here $\lambda'$ stands for the transposed diagram and $\nu = \nu(x)$.

Let $Y_k$ be a set of a Young diagrams $\lambda$ such that $|\lambda| = k$. We introduce further the set $Y_k(x) = \{ \lambda \in Y_k : \tilde{s}_\lambda (\alpha_\nu(x), 0) \neq 0 \}$ if $x > 0$, $\{ \lambda \in Y_k : \tilde{s}_\lambda (0, \beta_\nu(x)) \neq 0 \}$ if $x < 0$ and denote by $S_k^x(A_k)$ the set of all minimal projections in $C_k^x(A_k)$. By virtue of (3.22) and (3.24), the mapping

$$\lambda \in Y_k(x) \to \frac{\dim \lambda}{k!} E_{A_k}(x) \cdot \sum_{g \in S(A_k)} \chi(\lambda)(g) \pi_\chi(g) = e^T_k(\lambda) \in S_k^x(A_k)$$

is one-to-one correspondence and true.

**Lemma 3.7.** Let $S_k^x(\lambda)(A_k)$ be the set of all minimal projections in $e^T_k(\lambda)M^x_k(A_k)$. If $\epsilon \in S_k^x(\lambda)(A_k)$ then

$$\chi_{\alpha, \beta}^{(S)}(\epsilon) = [\nu(x)|x|^k] \cdot \left\{ \begin{array}{ll}
\tilde{s}_\lambda (\alpha_\nu(x), 0) & \text{if } x > 0, \\
\tilde{s}_\lambda (0, \beta_\nu(x)) & \text{if } x < 0
\end{array} \right. = o^\lambda_x(k)|x|^k,$$

where $o^\lambda_x(k) \in \mathbb{N}$ $\forall k \in \mathbb{N}$ and

$$\lim_{k \to \infty} \frac{o^\lambda_x(k)}{k^{\nu(x)}} = 0$$

uniformly on the set $Y_k$.

**Proof.** Using (3.22), (3.23) and (3.24), we obtain the statement of Lemma 3.7 from the next chain of equalities

$$\chi_{\alpha, \beta}^{(S)}(\epsilon) = \chi_{\alpha, \beta}^{(S)}(E_{A_k}(x)) \cdot \tau_{A_k,x}(\epsilon) = [\nu(x)|x|^k] \cdot \left\{ \begin{array}{ll}
\tilde{s}_\lambda (\alpha_\nu(x), 0) & \text{if } x > 0, \\
\tilde{s}_\lambda (0, \beta_\nu(x)) & \text{if } x < 0
\end{array} \right.$$

Let us denote for $x > 0$

$$o^\lambda_x(k) = \begin{vmatrix}
\nu + \lambda - 1 & \nu + \lambda & \ldots & 2 \nu + \lambda - 2 \\
\nu - 1 & \nu - 1 & \ldots & 2 \nu - 1 \\
\nu - 2 & \nu - 2 & \ldots & 2 \nu - 3 \\
\vdots & \ddots & \ddots & \vdots \\
\lambda & \lambda & \ldots & \lambda \\
\nu & \nu & \ldots & \nu
\end{vmatrix}$$

and for $x < 0$

$$o^\lambda_x(k) = \begin{vmatrix}
\nu + \lambda'_x - 1 & \nu + \lambda'_x & \ldots & 2 \nu + \lambda'_x - 2 \\
\nu - 1 & \nu - 1 & \ldots & 2 \nu - 1 \\
\nu - 2 & \nu - 2 & \ldots & 2 \nu - 3 \\
\vdots & \ddots & \ddots & \vdots \\
\lambda & \lambda & \ldots & \lambda \\
\nu & \nu & \ldots & \nu
\end{vmatrix}.$$
By means of $\tilde{m} \in \mathcal{N}_{\alpha,\beta} (m, k_\alpha, k_\beta, D)$ (see (3.11), p. 27) we introduce pairwise disjoint subsets $\mathcal{Q} (\alpha'_j) (1 \leq j \leq k_\alpha)$ and $\mathcal{Q} (\beta'_j) (1 \leq j \leq k_\beta)$ in $\mathcal{I}(m) = \{1, 2, \ldots, m\}$ with properties

$$|\mathcal{Q} (\alpha'_j)| = m_j^\alpha, \ |\mathcal{Q} (\beta'_j)| = m_j^\beta.$$ Let $\mathcal{Q}(\gamma) = \mathcal{I}(m) \setminus \left(\bigcup_{j=1}^{k_\alpha} \mathcal{Q}(\alpha'_j) \cup \bigcup_{j=1}^{k_\beta} \mathcal{Q}(\beta'_j)\right)$ and

$$E_{\tilde{Q}(\tilde{m})} = \left[ \prod_{j=1}^{k_\alpha} \prod_{l=1}^{k_\beta} \prod_{s \in \mathcal{Q}(\alpha'_j)} E_s (\alpha'_j) \prod_{s \in \mathcal{Q}(\beta'_j)} E_s (\beta'_j) \right] \prod_{s \in \mathcal{Q}(\gamma)} F_s, \quad (3.25)$$

where $F_s = \mathbb{I} - \sum_{j=1}^{k_\alpha} E_s (\alpha'_j) - \sum_{j=1}^{k_\beta} E_s (\beta'_j)$. $\tilde{Q}(\tilde{m})$ is an ordered set $(\mathcal{Q}(\alpha'_1), \ldots, \mathcal{Q}(\alpha'_k) ; \mathcal{Q}(\beta'_1), \ldots, \mathcal{Q}(\beta'_k) ; \mathcal{Q}(\gamma))$. If

$$G(\tilde{Q}(\tilde{m})) = \prod_{j=1}^{k_\alpha} \prod_{i=1}^{k_\beta} S (\mathcal{Q}(\alpha'_j) \times \mathcal{Q}(\beta'_i))$$

and $\mathcal{L}_\chi (G(\tilde{Q}(\tilde{m})))$ is generated by operators $\pi_\chi (G(\tilde{Q}(\tilde{m})))$ as a $W^*$-algebra, then $\mathcal{M}(\tilde{Q}(\tilde{m})) = E_{\tilde{Q}(\tilde{m})} \mathcal{L}_\chi (G(\tilde{Q}(\tilde{m})))$ is isomorphic to

$$\bigotimes_{j=1}^{k_\alpha} \mathcal{M}_{m_j}^{\alpha'_j} (\mathcal{Q}(\alpha'_j)) \bigotimes_{i=1}^{k_\beta} \mathcal{M}_{m_i}^{\beta'_i} (\mathcal{Q}(\beta'_i)) \quad (\text{see p.30}). \quad (3.26)$$

**Lemma 3.8.** Let $tr_\chi$ is the central normalize state on $\mathcal{L}_\chi (S (\infty))$ which corresponds to $\chi = \chi^{(S)}_{\alpha,\beta}$. If $\varphi_{\tilde{m}}$ is the restriction $tr_\chi$ to the algebra $\mathcal{M}(\tilde{Q}(\tilde{m}))$ and $H_{\varphi_{\tilde{m}}} (\mathcal{M}(\tilde{Q}(\tilde{m})))$ is the CNT-entropy of $\mathcal{M}(\tilde{Q}(\tilde{m}))$ corresponding to $\varphi_{\tilde{m}}$, then

$$H_{\varphi_{\tilde{m}}} (\mathcal{M}(\tilde{Q}(\tilde{m}))) = -\gamma_{k_\alpha k_\beta} \prod_{j=1}^{k_\alpha} a_j^{m_j^\alpha} \prod_{j=1}^{k_\beta} b_j^{m_j^\beta} \times \left\{ (m_{k_\alpha+k_\beta}+1) \ln (\gamma_{k_\alpha k_\beta}) + \sum_{j=1}^{k_\alpha} m_j^\alpha \ln \alpha'_j + \sum_{j=1}^{k_\beta} m_j^\beta \ln \beta'_j + O (\ln m) \right\},$$

where $0 \leq \limsup_{m \to \infty} \frac{O (\ln m)}{\ln m} < \infty$, $\gamma_{k_\alpha k_\beta} = \chi (F_s) = 1 - \sum_{j=1}^{k_\alpha} a_j - \sum_{j=1}^{k_\beta} b_j$.  

Proof. We denote by \( \varphi_{A_k}^x \) the restriction \( tr_X \) to the algebra \( \mathfrak{M}_k^x (A_k) \). In view of (3.25) and (3.26), we have

\[
\varphi_{\tilde{m}} = \gamma_{\alpha, \beta}^{m_{\alpha} + k_{\beta} + 1} \prod_{j=1}^{k_{\alpha}} \varphi(a'_j) \prod_{i=1}^{k_{\beta}} \varphi(b'_i).
\]

(3.27)

Let

\[
f^\lambda_x = \begin{cases} \tilde{s}_\lambda (\alpha_{\nu(x)}, 0) & \text{if } x > 0, \\ \tilde{s}_\lambda (0, \beta_{\nu(x)}) & \text{if } x < 0. \end{cases}
\]

Further, using (3.22) and (3.24), we obtain

\[
H_{\varphi_{A_k}^x} (\mathfrak{M}_k^x (A_k)) = - (\nu(x)|x|)^k \left\{ \sum_{\lambda: |\lambda| = k} \dim \lambda \cdot f^\lambda_x \left[ \ln f^\lambda_x + \ln \left( (\nu(x)|x|)^k \right) \right] \right\}
\]

\[
= - (\nu(x)|x|)^k \left\{ \sum_{\lambda: |\lambda| = k} \dim \lambda \cdot f^\lambda_x \left[ k \ln |x| + \ln \left( (\nu(x)^k f_x \right) \right] \right\}
\]

(see Lemma 3.7)

\[
= - (\nu(x)|x|)^k \left\{ \sum_{\lambda: |\lambda| = k} \dim \lambda \cdot f^\lambda_x \left[ k \ln |x| + \ln \left( \alpha_x^\lambda (k) \right) \right] \right\}
\]

Since \( \sum_{\lambda: |\lambda| = k} \dim \lambda \cdot f^\lambda_x = 1 \), we may rewrite \( H_{\varphi_{A_k}^x} (\mathfrak{M}_k^x (A_k)) \) as follows:

\[
H_{\varphi_{A_k}^x} (\mathfrak{M}_k^x (A_k)) = - k (\nu(x)|x|)^k \ln |x| - (\nu(x)|x|)^k \mathcal{O} (x, \ln k),
\]

(3.28)

where \( \mathcal{O} (x, \ln k) = \sum_{\lambda: |\lambda| = k} \dim \lambda \cdot f^\lambda_x \ln \left( \alpha_x^\lambda (k) \right) \) and by Lemma 3.7

\[
0 \leq \limsup_{k \to \infty} \frac{\mathcal{O} (x, \ln k)}{\ln k} < \infty.
\]

(3.29)

It follows from (3.26), (3.27) and (3.28) that

\[
H_{\varphi_{\tilde{m}}} (\mathcal{M} (\tilde{Q} (\tilde{m}))) = - \gamma_{\alpha, \beta}^{m_{\alpha} + k_{\beta} + 1} \prod_{j=1}^{k_{\alpha}} a_j^{m_j} \prod_{j=1}^{k_{\beta}} b_j^{m_j}
\]

\[
\times \left\{ (m_{\alpha} + k_{\beta} + 1) \ln \left( \gamma_{\alpha, \beta} \right) + \sum_{j=1}^{k_{\alpha}} m_j \ln \alpha_j' + \sum_{j=1}^{k_{\beta}} m_j \ln \beta_j' \right. 
\]

\[
+ \sum_{j=1}^{k_{\alpha}} \mathcal{O} (\alpha_j', \ln m_j^\alpha) + \sum_{j=1}^{k_{\beta}} \mathcal{O} (\beta_j', \ln m_j^\beta) \right\}.
\]
Using (3.29) and the equality \( \sum_{j=1}^{k_\alpha} m_j^\alpha + \sum_{j=1}^{k_\beta} m_j^\beta + m_{k_\alpha+k_\beta+1} = m \) in latter assertion, we have

\[
0 \leq \limsup_{m \to \infty} \frac{\sum_{j=1}^{k_\alpha} \mathcal{D}(\alpha_j', \ln m_j^\alpha) + \sum_{j=1}^{k_\beta} \mathcal{D}(\beta_j', \ln m_j^\beta)}{\ln m} < \infty
\]

and

\[
H_{\varphi_{\tilde{m}}} \left( \mathcal{M} \left( \tilde{\mathcal{Q}} (\tilde{m}) \right) \right) = -\gamma_{k_\alpha k_\beta} m_{k_\alpha+k_\beta+1} \prod_{j=1}^{k_\alpha} a_j \prod_{j=1}^{k_\beta} b_j
\times \left\{ (m_{k_\alpha+k_\beta+1}) \ln (\gamma_{k_\alpha k_\beta}) + \sum_{j=1}^{k_\alpha} m_j^\alpha \ln \alpha_j' + \sum_{j=1}^{k_\beta} m_j^\beta \ln \beta_j' + \mathcal{O}(\ln m) \right\}.
\]

In the next statements we will use the notations from Lemma 3.4.

**Proposition 3.1.** Let \( \mathcal{M}_m \) be a \( W^* \)-algebra generated by \( A_0 \) and \( \pi_\chi(S(m)) \) and let \( \delta_1, \delta_2 \) be given. Then there exists a natural numbers: \( k_\alpha (\delta_1), k_\beta (\delta_1), C (\delta_1, \delta_2), \) and \( M (\delta_1, \delta_2) \) such that

\[
i) \quad \gamma_{k_\alpha k_\beta} = 1 - \sum_{j=1}^{k_\alpha (\delta_1)} a_j - \sum_{j=1}^{k_\beta (\delta_1)} b_j < \delta_1;
\]

\[
ii) \quad \forall \ m > M (\delta_1, \delta_2)
\]

\[
H_\chi (\mathcal{M}_m) \geq -m (1 - \delta_2) \left\{ \sum_{j=1}^{k_\alpha (\delta_1)} \nu (\alpha_j') \alpha_j' \ln \alpha_j' + \sum_{j=1}^{k_\beta (\delta_1)} \nu (\beta_j') \beta_j' \ln \beta_j' + C (\delta_1, \delta_2) \sqrt{m} \right\} \quad \text{(see Lemma 3.8)}.
\]

**Proof.** It is clear that \( \mathcal{M} (\tilde{\mathcal{Q}} (\tilde{m})) \subset \mathcal{M}_m \). Therefore,

\[
H_\chi (\mathcal{M}_m) \geq \sum_{m} \frac{m! \cdot H_{\varphi_{\tilde{m}}} \left( \mathcal{M} \left( \tilde{\mathcal{Q}} (\tilde{m}) \right) \right)}{m^{k_\alpha (\delta_1) + k_\beta (\delta_1) + 1} \prod_{j=1}^{k_\alpha (\delta_1)} m_j^\alpha! \prod_{j=1}^{k_\beta (\delta_1)} m_j^\beta!}.
\]
First we recall the well-known construction (see [20]) of the embedding of the group. Proof.

(Lemma 3.8) \[
\geq -\left\{ \sum_{\alpha} m! \cdot \gamma_{k,\alpha(\Theta)} k_{\alpha}(\Theta) + k_{\beta}(\Theta) + 1 \right\} \prod_{j=1}^{k} a_j^{m_{\alpha}(\Theta)} b_j^{m_{\beta}(\Theta)}
\]

\[
\times \left[ \sum_{j=1}^{k_{\alpha}(\Theta)} m_{\alpha} \ln \alpha_j + \sum_{j=1}^{k_{\beta}(\Theta)} m_{\beta} \ln \beta_j \right] + O(\ln m) \}
\]

(Lemma 3.4, (3.11)) \[
\geq -m(1-\delta_2) \left[ \sum_{j=1}^{k_{\alpha}(\Theta)} a_j \ln \alpha_j + \sum_{j=1}^{k_{\beta}(\Theta)} b_j \ln \beta_j \right] + C(\delta_1, \delta_2) \sqrt{m}.
\]

The latter inequality is true for all sufficiently large \( m \).

**Proposition 3.2.** (An upper bound for the entropy) \( \eta(t) = -t \ln t \), \( \sum \alpha_i + \sum \beta_i = 1, \chi = \chi_{\alpha,\beta}^{(S)} \). Then

\[
H\chi(\theta) \leq \sum_j \eta(\alpha_j) + \sum_j \eta(\alpha_j).
\]

**Proof.** First we recall the well-known construction (see [20]) of the embedding of the group \( S(\infty) \) in the Powers factor. Let \( n(\alpha) = \min \{ i : \alpha_i > 0 \}, \ n(\beta) = \min \{ i : \beta_i > 0 \}, \ n = n(\alpha) + n(\beta), \ \mathcal{N}_n^\alpha = \{ 1, 2, \ldots, n(\alpha) \}, \ \mathcal{N}_n^\beta = \{ -1, -2, \ldots, -n(\beta) \} \) and \( \mathfrak{N}_n = \mathcal{N}_n^\alpha \cup \mathcal{N}_n^\beta. \) We consider the algebra \( M_n(\mathbb{C}) \) of all complex \( n \times n \)-matrices with system of matrix units \( \{ e_i,j \}_{i,j \in \mathfrak{N}_n} \). Let \( h_{\alpha\beta} = \text{diag}(\alpha_1, \ldots, \alpha_{n(\alpha)}; \beta_1, \ldots, \beta_{n(\beta)}) \in M_n(\mathbb{C}) \) and let \( \varphi(\cdot) = Tr(\cdot h_{\alpha\beta}) \) is the state on \( M_n(\mathbb{C}) \), where \( Tr \) is ordinary trace. For \( j \in \mathbb{Z} \) let \( \mathcal{M}_j = M_n(\mathbb{C}) \) and \( \varphi_j = \varphi \). Let \( (\mathcal{M}, \varphi) = \bigotimes_{j \in \mathbb{Z}} (\mathcal{M}_j, \varphi_j) \), where \( \hat{\varphi} = \bigotimes_{j \in \mathbb{Z}} \varphi_j. \) For sequence \( i_k = (i_j \in \mathfrak{N}_n)_{j=-k}^{j=k} \) let

\[
j(i_k) = \left( \{ j_1 < j_2 < \ldots < j_l(i_k) \} : i_{j_l} \in \mathcal{N}_n^\beta \ \forall l = 1, 2, \ldots, l(i_k) \right).
\]

If \( g \) is any permutation of the set \( B_k = \{ -k, \ldots, 0, \ldots, k \} \), then there is permutation \( s(g, i_k) \in S(g(j(i_k))) \) such that

\[
s(g, i_k)(g(j_1)) < s(g, i_k)(g(j_2)) < \ldots < s(g, i_k)(g(j_l(i_k))).
\]

Let \( \psi(g, i_k) = sgn(s(g, i_k)) \) and let \( \mathcal{I}_k \) be the set of all sequences \( (i_j \in \mathfrak{N}_n)_{j=-k}^{j=k}. \) Now by a direct checking we can make sure, that operators \( U_g = \sum_{i_k \in \mathcal{I}_k} \psi(g, i_k) e_{i_k} g(i_k) \), where

\[
e_{i_k} g(i_k) = e_{i_k} g(\cdot-k) \otimes e_{i_k} g(\cdot-k+1) \otimes \ldots e_{i_k} g(\cdot-k) \in \mathcal{M}_{\varphi},
\]
\(\mathcal{M}_\varphi\) is the centralizer of \(\varphi\), define a unitary representation of the group 
\(S(2k + 1) = S(B_k)\) and \(\varphi(U_g) = \chi^{(S)}_{\alpha,\beta}(g)\). Further, we notice that
automorphism \(\vartheta^\chi_S\) of the \(W^*\)-algebra generated by \(U_g\) \((g \in \bigcup_k S(B_k))\) ex-
tends to the automorphism \(\theta\) of the \(W^*\)-algebra \(\mathcal{M}\). But it is well-known
(see [5]), that \(\theta\) is a noncommutative Bernouli shift with the entropy
\[\sum_j \eta(\alpha_j) + \sum_j \eta(\beta_j).\]

Proof of Theorem 3.1. The statement of the theorem follows from Lemma
3.2 and Proposition 3.2 when sets \(\{\alpha_i \neq 0\}\) and \(\{\beta_i \neq 0\}\) are finite ones.
In the general case we consider the algebra \(\mathcal{M}_m\) that is generated by
\(A_0\) and \(\pi^\chi(S(m))\) as a \(W^*\)-algebra. Using a method by which we have
proved Lemma 2.1, we receive the following estimation
\[H^\chi(\vartheta^\chi_S) \geq \frac{H^\chi(\mathcal{M}_m)}{m}.\]
Hence, tacking into account Propositions 3.2 and 3.1, we complete the
proof. \(\square\)

References


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