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MODAL APPROACH TO THE CONTROLLABILITY PROBLEM OF DISTRIBUTED PARAMETER SYSTEMS

This paper is devoted to the analysis of the controllability for a class of linear control systems in a Hilbert space. It is proposed to use the minimum energy control of a reduced lumped parameter system for solving the infinite dimensional steering problem approximately. Sufficient conditions of the approximate controllability are formulated for a modal representation of a flexible structure with small dissipation.

Key words: *distributed parameter system, approximate controllability, modal analysis, spillover.*

1. Introduction. The problems of spectral, approximate, exact, and null controllability of distributed parameter systems have been intensively studied in the last few decades [2, 3, 4]. On the one hand, the question of the approximate controllability of a linear time-invariant system on a Hilbert space can be formulated in terms of an invariant subspace of the corresponding adjoint semigroup [5], [3, p. 56]. On the other hand, the problem of an effective control design remains challenging for a wide range of mechanical systems with distributed parameters.

The goal of this work is to propose a constructive control strategy, based on a reduced model, and to justify that this approach can be used to solve the approximate controllability problem for the whole infinite dimensional system.

2. Problem statement. This paper addresses the problem of approximate controllability of a linear differential equation

$$\dot{x} = Ax + Bu, \quad x \in H,\ u \in \mathbb{R}^m, \tag{1}$$

where H is a Hilbert space, $A : D(A) \to H$ is a closed densely defined operator, and $B : \mathbb{R}^m \to H$ is a continuous operator. We assume that A generates a strongly continuous semigroup of operators $\{e^{tA}\}_{t\geq 0}$ on H. Hence, for any $x^0 \in H$ and $u \in L^2(0,\tau)$, the mild solution of (1) corresponding to the initial condition $x|_{t=0} = x^0$ and control $u = u(t)$ can be written as follows

$$x(t; x^0, u) = e^{tA}x^0 + \int_0^t e^{(t-s)A}Bu(s)\, ds,\ 0 \leq t \leq \tau. \tag{2}$$

Recall that that system (1) is *approximately controllable in time* $\tau > 0$ if (cf. [3]), given $x^0, x^1 \in H$ and $\varepsilon > 0$, there exists $u \in L^2(0,\tau)$ such that

$$\|x(\tau; x^0, u) - x^1\| < \varepsilon.$$

In order to study the approximate controllability of system (1), we use the following result.

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Proposition 1. [7] *Let* $\{Q_N\}_{N=1}^{\infty}$ *be a family of bounded linear operators on* H *satisfying the following conditions:*

1)
$$\lim_{N\to\infty} \|Q_N x\| = 0, \quad \text{for all}\, x \in H; \tag{3}$$

2) the operators e^{tA} *and* Q_N *commute;*

3) for each $x^0, x^1 \in H$, $N \geq 1$, *there is a control* $u^N_{x^0,x^1} \in L^\infty(0,\tau)$ *such that*

$$(I - Q_N)\Big(x(\tau; x^0, u^N_{x^0,x^1}) - x^1\Big) = 0, \tag{4}$$

$$\lim_{N\to\infty}\Big(\|Q_N B\| \cdot \|u^N_{x^0,x^1}\|_{L^2(0,\tau)}\Big) = 0. \tag{5}$$

Then system (1) is approximately controllable in time τ, *and the above family of functions* $u = u^N_{x^0,x^1}(t)$, $0 \leq t \leq \tau$, *can be used to solve the approximate controllability problem.*

In this paper, I stands for the identity operator on H. For a possible application of the above proposition, we assume that each operator $P_N = I - Q_N$ is a finite dimensional projection.

Let $\dim(\operatorname{Im} P_N) = d_N$. For given $x^0, x^1 \in H$, we introduce vectors

$$\tilde{x}^0_N = P_N x^0,\ \tilde{x}^1_N = P_N x^1,\ \tilde{x}_N = P_N x, \quad (\tilde{x}^0_N, \tilde{x}^1_N, \tilde{x}_N \in \operatorname{Im} P_N),$$

and operators $\tilde{A}_N = P_N A$, $\tilde{B}_N = P_N B$. Then condition (4) implies that $u^N_{x^0,x^1}(t)$ should solve the following control problem:

$$\dot{\tilde{x}}_N = \tilde{A}_N \tilde{x}_N + \tilde{B}_N u, \quad t \in [0,\tau], \tag{6}$$
$$\tilde{x}_N|_{t=0} = \tilde{x}^0_N,\ \tilde{x}_N|_{t=\tau} = \tilde{x}^1_N.$$

Here we have used the assumption that P_N and A commute as well as the property $P_N = P_N^2$ of a projection. To satisfy condition (5), it is natural to look for a control $u = u^N_{x^0,x^1}(t)$ that minimizes the functional

$$J = \int_0^\tau (Qu, u)\, dt \to \min \tag{7}$$

with some symmetric positive definite $m \times m$-matrix Q. As control system (6) evolves on a real d_N-dimensional vector space $\operatorname{Im} P_N$, we may treat (6) as a system on $\mathbb{R}^{d_N}$ without lack of generality. By applying the Pontryagin maximum principle (see [1]), we get the optimal control for problem (6)-(7):

$$\tilde{u}(t) = Q^{-1}\tilde{B}'_N e^{(\tau-t)\tilde{A}'_N}\nu, \tag{8}$$

$$\nu = \left(\int_0^\tau e^{s\tilde{A}_N}\tilde{B}_N Q^{-1}\tilde{B}'_N e^{s\tilde{A}'_N}\, ds\right)^{-1} (\tilde{x}^1_N - e^{\tau\tilde{A}_N}\tilde{x}^0_N),$$

where the prime stands for the transpose. Proposition 1 implies that the proof of the approximate controllability can be reduced to the checking conditions (3) and (5) with a family of smooth controls $u^N_{x^0,x^1} = \tilde{u}(t)$ given by (8). The main contribution of this paper is the application of such a scheme for a class of systems (1) representing the oscillations of a flexible structure with damping.

3. Flexible system with damping. Consider a particular case of system (1) as follows

$$\dot{x} = Ax + Bu, \quad x \in \ell^2,\ u \in \mathbb{R}, \tag{9}$$

where

$$x = (\xi_0, \eta_0, \xi_1, \eta_1, \xi_2, \eta_2, ...)',$$

$$\|x\|^2_{\ell^2} = \sum_{n=0}^{\infty} (\xi_n^2 + \eta_n^2).$$

We assume that the operator $A : D(A) \to \ell^2$ in (9) is given by its block-diagonal matrix:

$$A = \operatorname{diag}(A_0, A_1, A_2, ...) = \begin{pmatrix} A_0 & 0 & 0 & \dots \\ 0 & A_1 & 0 & \dots \\ 0 & 0 & A_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_n = \begin{pmatrix} 0 & \omega_n \\ -\omega_n & -2\kappa \end{pmatrix}, \quad n = 1, 2, ... ,$$

and $B = (0, 1, 0, b_1, 0, b_2, ...)' \in \ell^2$. Control system (9) is a linear model of a rotating flexible beam attached to a rigid body. The components ξ_n and η_n of x are the modal coordinate and modal velocity corresponding to the mode number n. The control u is the angular acceleration of the body. Coefficients ω_n and b_n are, respectively, the modal frequency and the control coefficient corresponding to the n-th mode of oscillations of the beam. The coefficient $\kappa > 0$ represents the viscous damping in the beam. The procedure of deriving the equations of motion with modal coordinates is described in paper [6] for a rotating rigid body with several Euler-Bernoulli beams.

The main result of this paper is as follows.

Proposition 2. *Assume that $b_n \neq 0$ and $\omega_n > 0$ for all $n = 1, 2, ...$. Then there exists a $\tau > 0$ such that system (9) is approximately controllable in time τ provided that*

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} \frac{1}{(\omega_i - \omega_j)^2} < \infty \tag{10}$$

and that the damping coefficient κ is small enough.

Proof. Let us introduce the family of operators $P_N : \ell^2 \to \ell^2$ and $Q_N : \ell^2 \to \ell^2$ as follows:

$$P_N \begin{pmatrix} \xi_0 \\ \eta_0 \\ \vdots \\ \xi_N \\ \eta_N \\ \xi_{N+1} \\ \eta_{N+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \eta_0 \\ \vdots \\ \xi_N \\ \eta_N \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad Q_N = I - P_N, \quad N = 1, 2, \dots ,$$

where I is the identity operator on ℓ^2. Condition 1) of Proposition 1 holds with the above choice of Q_N. To check condition 2), we compute the semigroup $\{e^{tA}\}$ generated by A:

$$e^{tA} = \mathrm{diag}\left(e^{tA_0}, e^{tA_1}, e^{tA_2}, \dots\right),$$

$$e^{tA_0} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \; e^{tA_n} = e^{-\kappa t} \begin{pmatrix} \cos \beta_n t + \frac{\kappa}{\beta_n} \sin \beta_n t & \frac{\omega_n}{\beta_n} \sin \beta_n t \\ -\frac{\beta_n^2 + \kappa^2}{\beta_n \omega_n} \sin \beta_n t & \cos \beta_n t - \frac{\kappa}{\beta_n} \sin \beta_n t \end{pmatrix}, \; n \geq 1,$$

where $\beta_n = \sqrt{\omega_n^2 - \kappa^2} > 0$. We assume that the sequence $\{\omega_n\}$ is separated from zero and that the damping coefficient κ is small enough:

$$0 < \kappa < \inf_n \omega_n.$$

It is easy to see that condition 2) of Proposition 1 holds because of the block diagonal form of Q_N and e^{tA}.

For arbitrary $x^0, x^1 \in \ell^2$ and $N \geq 1$, we define the control $u^N_{x^0,x^1}(t) = \tilde{u}(t)$ by formula (8). In the case considered here,

$$m = \dim u = 1, \; Q = 1, \; \tilde{B}'_N = (0, 1, 0, b_1, \dots, 0, b_N),$$

$$\tilde{A}_N = \mathrm{diag}(A_0, A_1, \dots, A_N), \; e^{t\tilde{A}_N} = \mathrm{diag}(e^{tA_0}, e^{tA_1}, \dots, e^{tA_N}),$$
$$\tilde{x}^j_N = (\xi^j_0, \eta^j_0, \dots, \xi^j_N, \eta^j_N)', \quad j = 0, 1.$$

Then formula (8) takes the form

$$u^N_{x^0,x^1}(t) = \tilde{u}(t) = \tilde{B}'_N e^{(\tau - t)\tilde{A}'_N} \nu, \tag{11}$$

$$\nu = \left(\int_0^\tau M_N(s)\, ds \right)^{-1} (\tilde{x}^1_N - e^{\tau \tilde{A}_N} \tilde{x}^0_N),$$

where the matrix $M_N(s) = (M_{ij})^N_{i,j=0}$ is represented by its 2×2-blocks M_{ij} as follows

$$M_{00} = \begin{pmatrix} s^2 & s \\ s & 1 \end{pmatrix}, \; M_{0j} = \begin{pmatrix} S_j s & C_j s \\ S_j & C_j \end{pmatrix},$$

$$M_{i0} = \begin{pmatrix} S_i s & S_i \\ C_i s & C_i \end{pmatrix}, \; M_{ij} = \begin{pmatrix} S_i S_j & S_i C_j \\ C_i S_j & C_i C_j \end{pmatrix}, \quad (i,j = 1,2,...,N). \tag{12}$$

Here C_n and S_n are functions of s:

$$C_n = b_n e^{-\kappa s}\left(\cos\beta_n s - \frac{\kappa}{\beta_n}\sin\beta_n s\right),$$

$$S_n = \frac{b_n\omega_n}{\beta_n} e^{-\kappa s}\sin\beta_n s.$$

By exploiting formulae (12), we get the following representation of $\int_0^\tau M_N(s)\,ds$ for large τ:

$$H_N(\tau) = \int_0^\tau M_N(s)\,ds = \bar{B}_N \bar{W}_N \bar{B}_N + O(\tau e^{-\kappa\tau}), \tag{13}$$

where

$$\bar{B}_N = \mathrm{diag}(1,1,b_1,b_1,...,b_N,b_N)$$

and the 2×2-blocks W_{ij} of $\bar{W}_N = (W_{ij})_{i,j=0}^N$ are

$$W_{00} = \begin{pmatrix} \frac{\tau^3}{3} & \frac{\tau^2}{2} \\ \frac{\tau^2}{2} & \tau \end{pmatrix}, \quad W_{0j} = \begin{pmatrix} \frac{2\kappa}{\omega_j^3} & -\frac{1}{\omega_j^2} \\ \frac{1}{\omega_j} & 0 \end{pmatrix}, \quad W_{i0} = \begin{pmatrix} \frac{2\kappa}{\omega_i^3} & \frac{1}{\omega_i} \\ -\frac{1}{\omega_i^2} & 0 \end{pmatrix},$$

$$W_{ij} = \frac{1}{\{4\kappa^2 + (\beta_i-\beta_j)^2\}\{4\kappa^2+(\beta_i+\beta_j)^2\}}\begin{pmatrix} 4\kappa\omega_i\omega_j & \omega_i(\omega_i^2-\omega_j^2) \\ \omega_j(\omega_j^2-\omega_i^2) & 2\kappa(\omega_i^2+\omega_j^2) \end{pmatrix}, \; i,j \geq 1. \tag{14}$$

In particular, by using the identity $\beta_j^2 + \kappa^2 = \omega_j^2$, one obtains

$$W_{jj} = \frac{1}{4\kappa}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j \geq 1.$$

Let us denote

$$W_N = \mathrm{diag}(W_{00}, W_{11}, ..., W_{NN}), \quad F_N = \bar{W}_N - W_N. \tag{15}$$

It is easy to see that W_N is invertible for any $\tau > 0$ and $\kappa > 0$. Then a sufficient condition for the matrix $\bar{W}_N = W_N + F_N$ to be invertible is the condition of diagonal dominance:

$$\|F_N\| < \frac{1}{\|W_N^{-1}\|}. \tag{16}$$

Straightforward computations yield

$$\|W_N^{-1}\| = \max\left\{4\kappa, O\left(\frac{1}{\tau}\right)\right\} \quad \text{as } \tau \to +\infty. \tag{17}$$

The definition of F_N in (15) (and W_{ij} in (14)) implies that, under condition (10), there exists a constant $\bar{C} > 0$ such that

$$\|F_N\| \leq \bar{C}, \quad \text{for all } N = 1,2,\dots. \tag{18}$$

Estimates (17) and (18) mean that condition (16) holds if τ is large enough and $\kappa > 0$ is small enough. Hence, the matrix $H_N(\tau)$ in (13) is invertible for large τ. This implies that the control $u^N_{x^0,x^1}(t)$ in (11) is well-defined under our assumptions. As $u^N_{x^0,x^1}(t)$ solves finite-dimensional boundary value problem (6), condition (4) is satisfied for any $N \geq 1$.

By exploiting formula (11) together with representation (13) and the property

$$\sum_{n=1}^{\infty} b_n^2 < \infty,$$

we conclude that condition (5) holds for a dense in ℓ^2 set of initial (x^0) and terminal (x^1) points. Then system (9) is approximately controllable in ℓ^2 by Proposition 1. □

4. Conclusions. This work extends the result of [7] for the case of a flexible system with damping. As it was shown earlier in [7], condition (10) is satisfied for the Euler-Bernoulli beam without damping. Hence, condition (10) is sufficient for the approximate controllability in both conservative ($\kappa = 0$) and dissipative (small $\kappa > 0$) cases under our assumptions. An open question is whether is it possible to relax restriction (10) in order to justify the relevance of controls (8) under a weaker assumption on the distribution of the modal frequencies $\{\omega_n\}$.

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А.Л. Зуев

Модальный подход к проблеме управляемости систем с распределенными параметрами.

Статья посвящена анализу управляемости класса линейных систем управления в гильбертовом пространстве. Для приближенного решения бесконечномерной проблемы управляемости используется управление с минимальной энергией, которое соответствует подсистеме с сосредоточенными параметрами. Предложены достаточные условия приближенной управляемости системы с модальными координатами, которая описывает колебания упругой механической модели с малой диссипацией.

Ключевые слова: *система с распределенными параметрами, приближенная управляемость, модальный анализ, переливание энергии.*

О.Л. Зуєв

Модальний підхід до проблеми керованості систем з розподіленими параметрами.

Статтю присвячено аналізу керованості класу лінійних систем керування в гільбертовому просторі. Для наближенного розв'язання нескінченновимірної проблеми керованості використовується керування з мінімальною енергією, що відповідає підсистемі зі зосередженими параметрами. Запропоновано достатні умови наближеної керованості для системи з модальними координатами, що описує коливання пружної механічної моделі з малою дисипацією.

Ключові слова: *система з розподіленими параметрами, наближена керованість, модальний аналіз, переливання енергії.*

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