Spectral analysis of differential operators with indefinite weights and a local point interaction

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Abstract. We consider quasi-self-adjoint extensions of the symmetric operator $A = -(\operatorname{sgn} x)\frac{d^2}{dx^2}$, $\operatorname{dom}(A) = \{f \in W_2^2(\mathbb{R}) : f(0) = f'(0) = 0\}$, in the Hilbert space $L^2(\mathbb{R})$. The main result is a criterion of similarity to a normal operator for operators of this class. The spectra and resolvents of these extensions are described. As an application we describe the main spectral properties of the operators $(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + c\delta\right)$ and $(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + c\delta'\right)$.

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Introduction

Consider the symmetric operator A in the Hilbert space $L^2(\mathbb{R})$ defined by

$$\operatorname{dom}(A) = \{f \in W_2^2(\mathbb{R}) : f(0) = f'(0) = 0\},$$
$$(Af)(x) = -(\operatorname{sgn} x)f''(x) \qquad \text{for} \qquad f \in \operatorname{dom} A. \tag{0.1}$$

The object of investigation is the similarity of quasi–self-adjoint extensions of A (see [1]) to a normal operator. Let us recall that two operators T_1 and T_2 in a Hilbert space $\mathfrak{H}$ are called similar if there exists a bounded operator C with bounded inverse C^{-1} such that $T_1 = C^{-1}T_2C$.

Spectral problems

$$(Ly)(x) = \lambda r(x)y(x), \tag{0.2}$$

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where L is an elliptic operator and the function $r(x)$ change sign, occur in certain physical models (see [4] and references therein). The question whether the system of eigenfunctions of the problem (0.2) forms a Riesz basis was studied in [3], [4], [19], [32] [33], [34] (see also references in [34]). If the operator $\frac{1}{r}L$ has a nonempty continuous spectrum, then the corresponding problem is the similarity of $\frac{1}{r}L$ to a self-adjoint (normal) operator.

In [9], [6], [15], [7], [8], [16] the Krein–Langer spectral theory of definitizable operators (see [28]) was applied to similarity problems for quasi J-nonnegative operators (see [15]) of the form $\frac{1}{r}L$. In particular, B. Ćurgus and B. Najman [7] showed that the operator

$$\tilde{A} = -(\operatorname{sgn} x)\frac{d^2}{dx^2}, \qquad \operatorname{dom}(\tilde{A}) = W_2^2(\mathbb{R}), \tag{0.3}$$

is similar to a self-adjoint one.

This result was proved by another method in [21]; the method is based on the Naboko–Malamud criterion of similarity to a self–adjoint operator [31], [29] (see also [5]). One more proof is presented in [20]. In the recent papers [22], [13], [14], [24] the Naboko-Malamud criterion was applied to different J-self-adjoint differential operators.

Differential operators with an indefinite weight are of interest from one more point of view. The characteristic function $W(\cdot)$ of the operator $\frac{1}{r}L$ as well as the corresponding J-form $J - W^*JW$ is unbounded in $\mathbb{C}_+$. Therefore known sufficient conditions of similarity to a self–adjoint operator cannot be applied here (see [30], [20] and bibliography therein).

In the present paper we describe quasi-self-adjoint extensions A_B of the symmetric operator A in terms of boundary triplets (see [18], [11]). In Sections 3–4 we formulate a criterion of similarity of A_B to a normal (self–adjoint) operator. In order to illustrate these results in Section 5 we obtain simple similarity criteria for operators with local point interactions at zero

$$\tilde{A}_1 := \operatorname{sgn} x\left(-\frac{d^2}{dx^2} + c_1\delta\right),\ c_1 \in \mathbb{C}, \quad \tilde{A}_2 := \operatorname{sgn} x\left(-\frac{d^2}{dx^2} + c_2\delta'\right),\ c_2 \in \mathbb{C}.$$

(See definitions of the operators $\tilde{A}_1$, $\tilde{A}_2$ in [2] and also in Section 5 of the present paper).

The results of the paper were announced in [23].

Notation: By $\mathfrak{H}, \mathcal{H}$ we denote separable Hilbert spaces. The set of all bounded linear operators from $\mathfrak{H}$ to $\mathcal{H}$ is denoted by $[\mathfrak{H}, \mathcal{H}]$ or $[\mathfrak{H}]$ if $\mathfrak{H} = \mathcal{H}$. $\mathcal{C}(\mathfrak{H})$ stands for the set of closed densely defined operators in $\mathfrak{H}$. Let T be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows $\operatorname{dom}(T)$, $\ker(T)$, $\operatorname{ran}(T)$ are the domain, kernel, range of T, respectively. We denote by $\sigma(T)$, $\sigma_r(T)$, $\sigma_c(T)$ the point, residual and continuous spectra of T. By $\sigma_p(T)$ the set of eigenvalues of T is indicated. We denote the resolvent set by $\rho(T)$; $R_T(\lambda) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$, is the resolvent of T. Recall that $\sigma_r(T) = \{\lambda \in \sigma(T) \setminus \sigma_p(T) : \overline{\operatorname{ran}(T - \lambda I)} \neq \mathfrak{H}\}$, $\sigma_c(T) = \sigma(T) \setminus (\sigma_p(T) \bigcup \sigma_r(T))$.

We set $\mathbb{C}_\pm := \{\lambda \in \mathbb{C} \ : \ \pm \operatorname{Im} \lambda > 0\}$, $\mathbb{R}_+ := (0, +\infty)$, $\mathbb{R}_- := (-\infty, 0)$. By $\chi_{\mathcal{I}}(t)$ we denote the characteristic function of the interval $\mathcal{I}$, i.e., $\chi_{\mathcal{I}}(t) = 1$ for $t \in \mathcal{I}$, $\chi_{\mathcal{I}}(t) = 0$ for $t \notin \mathcal{I}$. Finally, we set $\chi_\pm(t) := \chi_{\mathbb{R}_\pm}(t)$.

1. Preliminaries

1.1. A similarity criterion

Our approach is based on the concept of boundary triplets (see [18], [11]) and the resolvent similarity criterion obtained by S. N. Naboko [31] and M. M. Malamud [29] (in [5] this criterion was obtained under an additional assumption).

Theorem 1.1 ([29, 31]). *A closed operator T in a Hilbert space $\mathfrak{H}$ is similar to a self-adjoint one if and only if $\sigma(A) \subset \mathbb{R}$ and for all $f \in \mathfrak{H}$ the inequalities*

$$\sup_{\varepsilon > 0} \int_{-\infty}^{+\infty} \varepsilon \left\| R_T\left(\mu + i\varepsilon\right) f \right\|^2 d\mu \leq C \left\| f \right\|^2 ,$$

$$\sup_{\varepsilon > 0} \int_{-\infty}^{+\infty} \varepsilon \left\| R_{T^*}\left(\mu + i\varepsilon\right) f \right\|^2 d\mu \leq C^* \left\| f \right\|^2 , \quad (1.1)$$

are valid with constants C and C^* independent of f.

1.2. Linear relations

Definition 1.1. (i) *A closed linear relation Θ in $\mathcal{H}$ is a closed subspace Θ of $\mathcal{H} \oplus \mathcal{H}$.*

(ii) *The closed linear relation Θ is symmetric if for all $\{f_1, g_1\}, \{f_2, g_2\} \in \Theta$ the condition*

$$(g_1, f_2) - (f_1, g_2) = 0, \quad (1.2)$$

is satisfied.

(iii) *The closed linear relation Θ is self-adjoint if it is maximal symmetric, i.e., Θ is symmetric and there does not exist a closed symmetric relation $\tilde{\Theta}$ such that Θ is properly contained in $\tilde{\Theta}$.*

Let us illustrate closed linear relations by simple examples.

Example 1.1. (i) *Let B be a closed operator in $\mathcal{H}$, not necessarily bounded. Then the graph $G(B)$ of B is a closed relation in $\mathcal{H}$. Moreover, if $B = B^*$ is a self–adjoint operator, then $G(B)$ is a self–adjoint relation in $\mathcal{H}$.*

(ii) *The subspaces $\Theta_0 := \{0\} \times \mathcal{H}$, $\quad \Theta_1 := \mathcal{H} \times \{0\}$ of $\mathcal{H} \times \mathcal{H}$ are self-adjoint relations in $\mathcal{H}$. Obviously, Θ_0 is not the graph of any operator.*

1.3. Boundary triplets

Let $A \in \mathcal{C}(\mathfrak{H})$ be a closed symmetric operator with equal deficiency indices $n_+(A) = n_-(A)$ ($n_\pm(T) := \dim \mathfrak{N}_{\pm i}$ and by $\mathfrak{N}_\lambda := \ker(T^* - \lambda)$ the deficiency subspaces of A are indicated). Without loss of generality we can assume that A is simple. This means that A has no self-adjoint parts.

Definition 1.2 ([1]). *(i) A closed extension $\tilde{A}$ of A is called a proper extension if $A \subset \tilde{A} \subset A^*$. The set of all proper extensions is denoted by Ext_A.*

(ii) A proper extension $\tilde{A}$ is called a quasi-self-adjoint if

$$\dim(\mathrm{dom}(\tilde{A})/\,\mathrm{dom}(A)) = n_\pm(A). \tag{1.3}$$

We recall the definition of a boundary triplet which may be considered as an abstract version of the second Green formula.

Definition 1.3 ([18]). *A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings*

$$\Gamma_j : \mathrm{dom}(A^*) \longrightarrow \mathcal{H}, \qquad j \in \{0, 1\}, \tag{1.4}$$

is called a boundary triplet for the adjoint operator A^* of A if the following two conditions are satisfied:

(i) The second Green's formula

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_\mathcal{H} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{H}, \qquad f, g \in \mathrm{dom}(A^*), \tag{1.5}$$

takes place and

(ii) the mapping

$$\Gamma : \mathrm{dom}(A^*) \longrightarrow \mathcal{H} \oplus \mathcal{H}, \qquad \Gamma f := \{\Gamma_0 f, \Gamma_1 f\}, \tag{1.6}$$

is surjective.

The above definition allows one to describe the set Ext_A in the following way (see [10, 11]).

Proposition 1.1 ([10, 11]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping Γ establishes a bijective correspondence $\tilde{A} \to \Theta := \Gamma(\mathrm{dom}(\tilde{A}))$ between the set Ext_A and the set of closed linear relations in $\mathcal{H}$.*

By Proposition 1.1 the following definition is natural.

Definition 1.4. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator A^*.*

(i) Denote $A_\Theta = \tilde{A}$ if $\Theta = \Gamma(\mathrm{dom}(\tilde{A}))$, that is

$$A_\Theta := A^* | D_\Theta, \qquad \text{where} \qquad D_\Theta := \{f \in \mathrm{dom}(A^*) : \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}. \tag{1.7}$$

(ii) If $\Theta = G(B)$ is the graph of $B \in \mathcal{C}(\mathcal{H})$, then $\mathrm{dom}(A_\Theta)$ is determined by the equation $\mathrm{dom}(A_B) = D_B := D_\Theta = \ker(\Gamma_1 - B\Gamma_0)$. We set $A_B := A_\Theta$.

Let us make the following remarks.

Remark 1.1. 1) *The deficiency indices $n_\pm(A)$ are equal to the dimension of $\mathcal{H}$, i.e., $\dim(\mathcal{H}) = n_\pm(A)$.*

2) *There exist two self-adjoint extensions $A_j := A^* | \ker(\Gamma_j)$ which are naturally associated to a boundary triplet. According to Definition 1.4 $A_j = A_{\Theta_j}, j \in \{0, 1\}$, where $\Theta_0 = \{0\} \times \mathcal{H}$, $\Theta_1 = \mathcal{H} \times \{0\}$. Conversely, if A_0 is a self-adjoint extension of A, then there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $A_0 = A^* | \ker(\Gamma_0)$.*

3) Θ *is the graph of an operator* $B \in \mathcal{C}(\mathcal{H})$ *iff* $\tilde{A}$ *and* A_0 *are disjoint, i.e.,* $\mathrm{dom}(\tilde{A}) \cap \mathrm{dom}(A_0) = \mathrm{dom}(A)$.

4) $\Theta = G(B)$ *with* $B \in [\mathcal{H}]$ *iff* $\tilde{A}$ *and* A_0 *are transversal, i.e.,* $\tilde{A}$ *and* A_0 *are disjoint and* $\mathrm{dom}(\tilde{A}) + \mathrm{dom}(A_0) = \mathrm{dom}(A^*)$.

Definition 1.5 ([12]). *The proper extension* $\tilde{A} \in Ext_A$ *is called almost solvable if there exists a boundary triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *and an operator* $B \in [\mathcal{H}]$ *such that*

$$\mathrm{dom}(\tilde{A}) = \mathrm{dom}(A_B) := \ker(\Gamma_1 - B\Gamma_0). \tag{1.8}$$

The set of almost solvable extensions is denoted by $\mathcal{A}s_A$. Note that the class $\mathcal{A}s_A$ is sufficiently wide. Proper extensions having two regular points $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\mathrm{Im}\,\lambda_1 \cdot \mathrm{Im}\,\lambda_2 < 0$ belong to $\mathcal{A}s_A$. All quasi-self-adjoint extensions are in $\mathcal{A}s_A$ if $n_\pm(A) < \infty$.

1.4. Weyl functions

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm–Liouville operators. In [10, 11] the concept of Weyl function was generalized to an arbitrary symmetric operator A with infinite deficiency indices $n_+(A) = n_-(A)$. In this subsection we recall basic facts about Weyl functions.

Definition 1.6 ([10, 11]). *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for the operator* A^*. *The Weyl function of* A *corresponding to the boundary triplet* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a unique mapping*

$$M(\cdot) : \rho(A_0) \longrightarrow [\mathcal{H}] \tag{1.9}$$

satisfying

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda \qquad \text{for all} \quad f_\lambda \in \mathfrak{N}_\lambda, \quad \lambda \in \rho(A_0), \tag{1.10}$$

where $\mathfrak{N}_\lambda := \ker(A^* - \lambda I)$.

It is well known (see [10, 11]) that the above implicit definition of the Weyl function is correct and $M(\cdot)$ is an R-function obeying $0 \in \rho(\mathrm{Im}(M(i)))$. The Weyl function immediately provides some information about the “spectral properties” of proper extensions. We confine ourselves to the case of almost solvable extensions of the symmetric operator A.

Proposition 1.2 ([11, 12]). *Suppose that* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^*, $M(\cdot)$ *is the corresponding Weyl function,* $\lambda \in \rho(A_0)$ *and* $B \in [\mathcal{H}]$. *Then:*

1) $\lambda \in \rho(A_B)$ *if and only if* $0 \in \rho(B - M(\lambda))$;

2) $\lambda \in \sigma_i(A_B)$ *if and only if* $0 \in \sigma_i(B - M(\lambda))$, $i \in \{p, r, c\}$.

1.5. γ-fields

With each boundary triplet we can associate a so-called γ-field.

Definition 1.7 ([11]). *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*. *The* γ*-field* $\gamma(\cdot)$ *corresponding to* Π *is defined by*

$$\gamma(\lambda) := (\Gamma_0 | \mathfrak{N}_\lambda)^{-1} : \mathcal{H} \longrightarrow \mathfrak{N}_\lambda, \quad \lambda \in \rho(A_0). \tag{1.11}$$

One can easily check that

$$\gamma(\lambda) = (A_0 - \lambda_0)(A_0 - \lambda)^{-1}\gamma(\lambda_0), \quad \lambda, \lambda_0 \in \rho(A_0), \tag{1.12}$$

and consequently $\gamma(\cdot)$ is a γ-field in the sense of [26]. It is shown in [11] that the γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ are related by

$$M(\lambda) - M(\lambda_0)^* = (\lambda - \bar{\lambda}_0)\gamma(\lambda_0)^*\gamma(\lambda), \quad \lambda, \lambda_0 \in \rho(A_0). \tag{1.13}$$

The relation (1.13) means the $M(\cdot)$ is a Q-function in the sense of [26].

The following version of the Krein-Naimark formula for canonical resolvents (see for instance [26]) is based on the notion of boundary triplets.

Theorem 1.2 ([10, 11]). *Let $\tilde{A}$ be an almost solvable extension of A ($\tilde{A} \in \mathcal{A}s_A$), i.e., $\tilde{A} = A_B$ with $B \in [\mathcal{H}]$ for some boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$. Then*

$$(A_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(B - M(\lambda))^{-1}\gamma^*(\overline{\lambda}), \qquad \lambda \in \rho(A_B). \tag{1.14}$$

Here $M(\cdot)$ and $\gamma(\cdot)$ are the Weyl function and γ–field corresponding to the triplet Π.

2. Extensions of the minimal operator

2.1. Boundary conditions

Consider the operator A of the form (0.1). It is obvious that A is a closed simple symmetric operator with deficiency indices $n_\pm(A) = 2$.

We denote by $\sqrt{z}$ the branch of the multifunction on the complex plane $\mathbb{C}$ with the cut along $\mathbb{R}_-$, singled out by the condition $\sqrt{-1 + i0} = i$.

Theorem 2.1. (*i*) *The adjoint operator A^* has the form*

$$A^* = -(\operatorname{sgn} x)\frac{d^2}{dx^2}, \qquad \operatorname{dom}(A^*) = W_2^2(\mathbb{R} \setminus \{0\}) := W_2^2(\mathbb{R}_-) \oplus W_2^2(\mathbb{R}_+). \tag{2.1}$$

(*ii*) *Let mappings $\Gamma_j : W_2^2(\mathbb{R} \setminus \{0\}) \to \mathbb{C}^2$, $j = \{0, 1\}$, be given by*

$$\Gamma_0 f = \begin{pmatrix} f(+0) \\ f'(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f'(+0) \\ -f(-0) \end{pmatrix}. \tag{2.2}$$

Then $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*.

(*iii*) *The corresponding Weyl function $M(\cdot)$ is*

$$M(\lambda) := \begin{pmatrix} -\sqrt{-\lambda} & 0 \\ 0 & -1/\sqrt{\lambda} \end{pmatrix}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.3}$$

(*iv*) *The corresponding γ–field $\gamma(\lambda) : \mathbb{C}^2 \to \mathfrak{N}_\lambda$ is*

$$\gamma(\lambda)\begin{pmatrix} c_+ \\ c_- \end{pmatrix} := c_+ \cdot e^{-\sqrt{-\lambda}x}\chi_+(x) + \frac{c_-}{\sqrt{\lambda}} \cdot e^{\sqrt{\lambda}x}\chi_-(x), \qquad c_\pm \in \mathbb{C}. \tag{2.4}$$

Proof. The first statement is obvious. Moreover, we have

$$(A^*f,\ g)-(f,\ A^*g)=f'(+0)\overline{g(+0)}+f'(-0)\overline{g(-0)}- \\ -f(+0)\overline{g'(+0)}-f(-0)\overline{g'(-0)},\quad f,\ g\in\operatorname{dom}(A^*). \tag{2.5}$$

Hence (ii) follows from Definition 1.3.

Note that

$$\mathfrak{N}_\lambda=\{f_\lambda(x):=c_+\cdot e^{-\sqrt{-\lambda}x}\chi_+(x)+c_-\cdot e^{\sqrt{\lambda}x}\chi_-(x):c_\pm\in\mathbb{C}\}. \tag{2.6}$$

Combining (2.2) and (2.6), one gets

$$\Gamma_0 f_\lambda=\begin{pmatrix} c_+ \\ c_-\sqrt{\lambda}\end{pmatrix},\qquad \Gamma_1 f_\lambda=\begin{pmatrix} -c_+\sqrt{-\lambda} \\ -c_-\end{pmatrix}. \tag{2.7}$$

By Definitions 1.6 and 1.7, we easily obtain (2.3) and (2.4). □

Let us introduce the following boundary conditions at zero

$$\begin{cases} a_{11}f(-0)+a_{12}f'(-0)+a_{13}f(+0)+a_{14}f'(+0)=0 \\ a_{21}f(-0)+a_{22}f'(-0)+a_{23}f(+0)+a_{24}f'(+0)=0\end{cases},\qquad a_{ij}\in\mathbb{C}. \tag{2.8}$$

By Definition 1.2, a quasi-self-adjoint extension $\tilde{A}$ of the operator A has the form

$$\tilde{A}=A_{(a_{ij})}=A^*|\operatorname{dom}(A_{(a_{ij})}), \\ \operatorname{dom}(A_{(a_{ij})})=\{f\in W_2^2(\mathbb{R}\setminus\{0\}):\ f\text{ satisfies conditions (2.8)}\}, \tag{2.9}$$

with the matrix

$$(a_{ij}):=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24}\end{pmatrix}$$

such that

$$\operatorname{rank}(a_{ij})=n_\pm(A)=2. \tag{2.10}$$

Consider three cases.

1) Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$ or $\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$. Then $\tilde{A}=A_-\oplus A_+$, where $A_\pm$ is an operator in $L^2(\mathbb{R}_\pm)$. By condition (2.10), we see that one of the operators A_-, A_+ is a symmetric with deficiency indices (1,1) and another one is an adjoint to a symmetric operator with deficiency indices (1,1). Hence $\mathbb{C}\setminus\mathbb{R}\subset\sigma_p(\tilde{A})$ and $\tilde{A}$ is not similar to a normal operator.

2) Suppose that $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$ and $\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24}\end{pmatrix}$ have a zero column. Since $\operatorname{rank}(a_{ij})=2$, it follows that $\tilde{A}=A_-\oplus A_+$, where A_+ and A_- are self–adjoint operators in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively. Thus $\tilde{A}=\tilde{A}^*$.

3) Suppose that there are three nonzero columns in (a_{ij}). In this case one of the determinants

$$\Delta_1 = \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix},$$
$$\Delta_3 = \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \tag{2.11}$$

does not vanish.

Evidently, only the case (3) is of interest to us.

2.2. The case $\Delta_1 \neq 0$

Let $\Delta_1 \neq 0$. (The cases $\Delta_2 \neq 0$, $\Delta_3 \neq 0$, and $\Delta_4 \neq 0$ will be considered in Section 4.) Then conditions (2.8) take the form

$$\begin{cases} f'(+0) = b_{11} f(+0) + b_{12} f'(-0) \\ -f(-0) = b_{21} f(+0) + b_{22} f'(-0). \end{cases} \tag{2.12}$$

Hence, by Definition 1.5, $A_{(a_{ij})} = A_B = A^* |\ker(\Gamma_1 - B\Gamma_0)$. Here

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

and the boundary triplet $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is of the form (2.2).

In what follows A_B stands for the operator

$$A_B := -\operatorname{sgn} x \frac{d^2}{dx^2}, \qquad \operatorname{dom}(A_B) = \{f \in W_2^2(\mathbb{R}\backslash\{0\}) : f \text{ satisfies } (2.12)\}. \tag{2.13}$$

For $B \in \mathbb{C}^{2\times 2}$ and the Weyl function $M(\cdot)$ of the form (2.3) we define the function $\varphi_B(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ by

$$\varphi_B(\lambda) := \det(B - M(\lambda)), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.14}$$

Lemma 2.1. *Suppose A_B is the operator of the form* (2.13) *and $|b_{12}| + |b_{21}| \neq 0$; then:*

(i) $\sigma_c(A_B) = \mathbb{R}$;
(ii) $\sigma_r(A_B) = \emptyset$;
(iii) $\sigma_p(A_B) = \{\lambda \in \mathbb{C}\setminus\mathbb{R} : \varphi_B(\lambda) = 0\} = \{\lambda \in \mathbb{C}_+ : \varphi_B(\lambda) = 0\} \cup \{\lambda \in \mathbb{C}_- : \varphi_{B^*}(\overline{\lambda}) = 0\}$.

Proof. Simple calculations show that there are no eigenvalues on the real axis and $\sigma_c(A_B) = \mathbb{R}$. The second and the third statements evidently follow from Proposition 1.2. □

Lemma 2.2. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet of the form* (2.2)*. Then*

$$\gamma^*(\overline{\lambda}) f = \begin{pmatrix} \int\limits_0^{+\infty} f(t) e^{-\sqrt{-\lambda} t} dt \\ \frac{1}{\sqrt{\lambda}} \int\limits_{-\infty}^{0} f(t) e^{\sqrt{\lambda} t} dt \end{pmatrix}, \qquad f \in L^2(\mathbb{R}). \tag{2.15}$$

Proof. By Theorem 2.1.(iv), $\gamma^*(\cdot)$ is a map from $L^2(\mathbb{R})$ to $\mathbb{C}^2$. To find $\gamma^*(\cdot)$ we use the equation

$$(\gamma(\lambda)c, f)_{L^2(\mathbb{R})} = (c, \gamma^*(\lambda)f)_{\mathbb{C}^2}, \qquad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{C}^2, \quad f \in L^2(\mathbb{R}). \tag{2.16}$$

Combining (2.4) and (2.16), one gets

$$c_1 \cdot \int_0^{+\infty} \overline{f(t)} e^{-\sqrt{-\lambda}t} dt + \frac{c_2}{\sqrt{\lambda}} \cdot \int_{-\infty}^0 \overline{f(t)} e^{\sqrt{\lambda}t} dt = c_1 \cdot \overline{(\gamma^*(\lambda)f)_1} + c_2 \cdot \overline{(\gamma^*(\lambda)f)_2}. \tag{2.17}$$

Hence (2.15) immediately follows from (2.17). □

Let us denote

$$y_+(f,\lambda) := \int_0^{+\infty} f(t) e^{-\sqrt{-\lambda}t} dt, \qquad y_-(f,\lambda) := \int_{-\infty}^0 f(t) e^{\sqrt{\lambda}t} dt, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.18}$$

Lemma 2.3. *Let the operator A_B be of the form* (2.13) *and $A_0 = A^*|\ker \Gamma_0$. Then*

$$\begin{aligned} &\left((A_B - \lambda)^{-1} f - (A_0 - \lambda)^{-1} f\right)(x) = \\ &= \frac{e^{-\sqrt{-\lambda}x} \cdot \chi_+(x)}{\varphi_B(\lambda)} \left(\left(b_{22} + \frac{1}{\sqrt{\lambda}} \right) y_+(f,\lambda) - \frac{b_{12}}{\sqrt{\lambda}} \cdot y_-(f,\lambda) \right) + \\ &+ \frac{e^{\sqrt{\lambda}x} \cdot \chi_-(x)}{\sqrt{\lambda} \cdot \varphi_B(\lambda)} \left(\frac{b_{11} + \sqrt{-\lambda}}{\sqrt{\lambda}} \cdot y_-(f,\lambda) - b_{21} \cdot y_+(f,\lambda) \right), \\ &\qquad f \in L^2(\mathbb{R}), \qquad \lambda \in \rho(A_B). \end{aligned} \tag{2.19}$$

Here $\varphi_B(\cdot)$ and $y_\pm(f,\cdot)$ are given by (2.14) *and* (2.18), *respectively.*

Proof. By (2.3), for $\lambda \in \rho(A_B)$ (see Lemma 2.1) one obtains

$$\begin{aligned} (B - M(\lambda))^{-1} &= \begin{pmatrix} b_{11} + \sqrt{-\lambda} & b_{12} \\ b_{21} & b_{22} + 1/\sqrt{\lambda} \end{pmatrix}^{-1} = \\ &= \frac{1}{\varphi_B(\lambda)} \begin{pmatrix} b_{22} + 1/\sqrt{\lambda} & -b_{12} \\ -b_{21} & b_{11} + \sqrt{-\lambda} \end{pmatrix}. \end{aligned} \tag{2.20}$$

Combining (2.4), (2.15), (2.20) with formula (1.14), we get (2.19). □

3. Similarity to a normal operator

3.1. The main result

For each $B \in \mathbb{C}^{2\times 2}$ let us define the function $\varphi_B^+ : \overline{\mathbb{C}_+} \to \overline{\mathbb{C}}$ in the following way. We set

$$\varphi_B^+(\lambda) := \varphi_B(\lambda) = \det(B - M(\lambda)) \qquad \text{for} \quad \lambda \in \mathbb{C}_+, \tag{3.1}$$

and for $x \in \overline{\mathbb{R}}$ by $\varphi_B^+(x)$ we denote the boundary values of $\varphi_B(\lambda)$ in $\mathbb{C}_+$,

$$\varphi_B^+(x) := \lim_{\substack{z \to x \\ z \in \mathbb{C}_+}} \det(B - M(z)), \qquad x \in \mathbb{R} \cup \{\infty\}. \tag{3.2}$$

Note that the function φ_B^+ is analytic on $\mathbb{C}_+$ and continuous on $\overline{\mathbb{C}_+} \setminus \{0\}$.

The following similarity criterion is the main result of the paper.

Theorem 3.1 (Main Theorem). *Assume that $\Delta_1 \neq 0$ and the operator A_B is defined by* (2.13). *Let φ_B^+ and $\varphi_{B^*}^+$ be the functions defined in* (3.1)–(3.2) *and $|b_{12}|+|b_{21}| \neq 0$. Then A_B is similar to a normal operator if and only if the following conditions hold:*

(i) φ_B^+ and $\varphi_{B^*}^+$ have no zeroes in the set $\mathbb{R} \cup \{\infty\}$;

(ii) φ_B^+ and $\varphi_{B^*}^+$ have no zeroes of the second order in $\mathbb{C}_+$.

Remark 3.1. *Suppose that $|b_{12}| + |b_{21}| = 0$. Then the operator A_B has the form*

$$A_B = A_- \oplus A_+,$$

where the operators $A_\pm : L^2(\mathbb{R}_\pm) \to L^2(\mathbb{R}_\pm)$ are given by

$$A_\pm := \mp \frac{d^2}{dx^2}, \qquad \mathrm{dom}(A_\pm) = \{f \in W_2^2(\mathbb{R}_\pm) : f(\pm 0) + b_\pm \cdot f'(\pm 0) = 0\}. \tag{3.3}$$

Here $b_+ := -1/b_{11}$, $b_- := b_{22}$. Operators $A_\pm$ are well studied.

Remark 3.2. *The function φ_B^+ has a simple form. Indeed, by* (2.14) *and* (2.3)*, we have*

$$\varphi_B^+(\lambda) = -ib_{22}\sqrt{\lambda} + (b_{11}b_{22} - b_{12}b_{21}) - i + \frac{b_{11}}{\sqrt{\lambda}}, \qquad \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}. \tag{3.4}$$

So the conditions of Theorem 3.1 *can be easily checked (see Section* 5*). Let us remark that the function φ_B^+ has at most two zeroes (a zero of multiplicity k is counted as k zeroes).*

A criterion of similarity to a self–adjoint operator immediately follows from Theorem 3.1.

Theorem 3.2. *Let $|b_{12}|+|b_{21}| \neq 0$. Then the operator A_B is similar to a self–adjoint one iff the functions φ_B^+ and $\varphi_{B^*}^+$ do not vanish in $\overline{\mathbb{C}_+}$.*

Proof. By Lemma 2.1.(iii), $\sigma(A_B) = \mathbb{R}$ iff the functions φ_B^+ and $\varphi_{B^*}^+$ have no zeroes in $\mathbb{C}_+$. Combining this fact with Theorem 3.1, we get Theorem 3.2. □

To prove the main theorem we recall the following Lemma.

Lemma 3.1. *If an operator T is similar to a normal one, then the inequality*

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{\mathrm{dist}(\lambda,\ \sigma(T))} \tag{3.5}$$

holds with some constant $C > 0$.

3.2. Some estimates

We start with the following lemma.

Lemma 3.2. *Let* $|b_{12}| + |b_{21}| \neq 0$. *Suppose there exists* $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ *such that* $\varphi_B^+(\lambda_0) = 0$ *or* $\varphi_{B^*}^+(\lambda_0) = 0$. *Then the operator* A_B *of the form* (2.13) *is not similar to a normal operator.*

Proof. Without loss of generality suppose that $b_{21} \neq 0$ and $\varphi_B^+(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{R}$.

It is obvious that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\sup_{\|f\|\leq 1} |y_+(f,\lambda)|^2 = \sup_{\|f\|\leq 1} \left| \int_{\mathbb{R}_+} f(t) e^{-\sqrt{-\lambda} t} dt \right|^2 = \\ = \|e^{-\sqrt{-\lambda}x} \chi_+(x)\|_{L^2}^2 = \frac{1}{|2 \operatorname{Re} \sqrt{-\lambda}|} = \frac{1}{|2 \operatorname{Im} \sqrt{\lambda}|}. \quad (3.6)$$

Further, we set $f_+(\cdot) := f(\cdot)\chi_+(\cdot)$, $f \in L^2(\mathbb{R})$. By (2.19), we have

$$\left\|(A_B - \lambda I)^{-1} f_+ - (A_0 - \lambda I)^{-1} f_+\right\|_{L^2}^2 = \\ = \left\| \frac{e^{-\sqrt{-\lambda}x} \cdot \chi_+(x)}{\varphi_B^+(\lambda)} \left(b_{22} + \frac{1}{\sqrt{\lambda}} \right) y_+(f_+,\lambda) \right\|_{L_2}^2 + \\ + |b_{21}| \cdot \left\| \frac{e^{\sqrt{\lambda}x} \cdot \chi_-(x)}{\sqrt{\lambda} \cdot \varphi_B^+(\lambda)} \cdot y_+(f_+,\lambda) \right\|_{L_2}^2, \\ \lambda \in \rho(A_B) \cap \mathbb{C}_+. \quad (3.7)$$

Combining (3.6) and (3.7), one obtains

$$\left\|(A_B - \lambda I)^{-1} - (A_0 - \lambda I)^{-1}\right\|_{L^2}^2 \geq \left| \left(b_{22} + \frac{1}{\sqrt{\lambda}} \right) \frac{1}{2\varphi_B^+(\lambda) \cdot \operatorname{Im} \sqrt{\lambda}} \right|^2 + \\ + \left| \frac{b_{21}}{4\sqrt{\lambda} \cdot \varphi_B^+(\lambda)} \right|^2 \cdot \frac{1}{|\operatorname{Re} \sqrt{\lambda} \cdot \operatorname{Im} \sqrt{\lambda}|}. \quad (3.8)$$

Now if we recall Lemma 2.1, we obtain that $\operatorname{dist}(\lambda, \sigma(A_B)) = |\operatorname{Im} \lambda|$ in some neighborhood of λ_0. Therefore, for sufficiently small ε and $\lambda = \lambda_0 + i\varepsilon$

$$\operatorname{dist}(\lambda, \sigma(A_B))^2 \cdot \left\|(A_B - \lambda I)^{-1} - (A_0 - \lambda I)^{-1}\right\|_{L^2}^2 \geq \\ \geq \left| \frac{b_{21} \cdot \operatorname{Im} \lambda}{4\sqrt{\lambda} \cdot \varphi_B^+(\lambda)} \right|^2 \cdot \frac{1}{|\operatorname{Re} \sqrt{\lambda} \cdot \operatorname{Im} \sqrt{\lambda}|} \geq C_1 \cdot \frac{1}{|\operatorname{Im} \lambda|}. \quad (3.9)$$

Then the left part of inequality (3.9) is unbounded in the neighborhood of λ_0. Note that A_0 is a self–adjoint operator. Hence inequality (3.5) is valid for A_0. Therefore, the function

$$\operatorname{dist}(\lambda, \sigma(A_B)) \cdot \left\| (A_B - \lambda I)^{-1} \right\|_{L^2} \tag{3.10}$$

is unbounded in the neighborhood of λ_0. By Lemma 3.1, the operator A_B is not similar to a normal operator.

If $\varphi_B^+(\infty) = 0$, then formula (3.4) implies $\varphi_B^+(\lambda) = b_{11}/\sqrt{\lambda}$ for $\lambda \in \mathbb{C}_+$. Hence for ε large enough

$$\operatorname{dist}(i\varepsilon, \sigma(A_B))^2 \cdot \left\| (A_B - i\varepsilon I)^{-1} - (A_0 - i\varepsilon I)^{-1} \right\|_{L^2}^2 \geq$$
$$\geq \frac{|b_{21}|}{4|b_{11}|} \cdot \frac{\varepsilon^2}{|\operatorname{Re}\sqrt{i\varepsilon} \cdot \operatorname{Im}\sqrt{i\varepsilon}|} = C_2 \varepsilon \ . \tag{3.11}$$

Since the right part of (3.11) is unbounded in $\mathbb{C}_+$, we see that A_B is not similar to a normal operator. □

Lemma 3.3. *Let $|b_{12}| + |b_{21}| \neq 0$. Suppose that the function φ_B^+ has a zero of algebraic multiplicity 2 in $\mathbb{C}_+$. Then A_B is not similar to a normal operator.*

Proof. Let $b_{21} \neq 0$ (the case $b_{12} \neq 0$ can be considered in the same way). Suppose $\lambda_0 \in \mathbb{C}_+$ is a zero of multiplicity 2 of $\varphi_B^+(\cdot)$. By (3.8), we have for $\lambda \in \rho(A_B) \cap \mathbb{C}_+$

$$\left\| (A_B - \lambda I)^{-1} - (A_0 - \lambda I)^{-1} \right\|_{L^2} \geq \left| \frac{b_{21}}{4\sqrt{\lambda} \cdot \varphi_B^+(\lambda)} \right| \cdot \frac{1}{|\operatorname{Re}\sqrt{\lambda} \cdot \operatorname{Im}\sqrt{\lambda}|^{1/2}}. \tag{3.12}$$

Since $\lambda_0 \in \rho(A_0)$, we see that (3.12) implies

$$\left\| (A_B - \lambda I)^{-1} \right\|_{L^2} \geq \frac{C_{\lambda_0}}{|\varphi_B^+(\lambda)|}, \qquad C_{\lambda_0} = const > 0, \tag{3.13}$$

in some neighborhood of λ_0. Therefore λ_0 is a pole of multiplicity 2 of the resolvent $(A_B - \lambda I)^{-1}$. Consequently, the operator A_B is not similar to a normal one. □

We also need the following estimates.

Lemma 3.4. *Let $\lambda = \mu + i\varepsilon$, $(\varepsilon > 0)$. Let $y_\pm(f, \lambda)$ be of the form (2.18). Then the following inequalities*

$$\varepsilon \int_{-\infty}^{+\infty} \left\| \frac{y_\pm(f,\lambda)}{\sqrt{\lambda}} \cdot e^{-\sqrt{-\lambda}x} \chi_+(x) \right\|_{L^2}^2 d\mu \leq 2\pi \cdot C_1 \|f\|_{L^2}^2, \tag{3.14}$$

$$\varepsilon \int_{-\infty}^{+\infty} \left\| \frac{y_\pm(f,\lambda)}{\sqrt{\lambda}} \cdot e^{\sqrt{\lambda}x} \chi_-(x) \right\|_{L^2}^2 d\mu \leq 2\pi \cdot C_2 \|f\|_{L^2}^2 \tag{3.15}$$

are valid for all $f \in L^2(\mathbb{R})$ with constants C_1, C_2 independent of ε and f.

Proof. Let us prove the inequality (3.14) for $y_-(f,\lambda)$.

Put $f_-(\cdot) := f(\cdot)\chi_-(\cdot),\ f \in L^2(\mathbb{R})$. Denote by $F(z)$ the Fourier transform of f_-,

$$F(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_-(t)e^{-izt}dt, \qquad z \in \mathbb{C}_+. \tag{3.16}$$

Note that $F(\cdot) \in H^2(\mathbb{C}_+)$ and $\|F\|_{H^2} = \|f_-\|_{L^2} \le \|f\|_{L^2}$. Further, we obtain

$$\int_{-\infty}^{+\infty} \left\| \frac{y_-(f,\lambda)}{\sqrt{\lambda}} \cdot e^{-\sqrt{-\lambda}x}\chi_+(x)\right\|_{L^2}^2 d\mu = 2\pi \int_{-\infty}^{+\infty} \frac{1}{|\lambda|}|F(i\sqrt{\lambda})|^2\|e^{-\sqrt{-\lambda}x}\|_{L^2}^2 d\mu =$$

$$= 2\pi \int_{-\infty}^{+\infty} \frac{1}{2|\sqrt{\lambda}\,\mathrm{Im}\,\sqrt{\lambda}|} \left|F(i\sqrt{\lambda})\right|^2 \frac{d\mu}{2\sqrt{|\lambda|}} \le$$

$$\le 2\pi \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \frac{1}{2|\,\mathrm{Im}\,\sqrt{\lambda}\,\mathrm{Re}\,\sqrt{\lambda}|} \left|F(i\sqrt{\lambda})\right|^2 |d\sqrt{\lambda}| =$$

$$= 2\pi \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \frac{1}{\mathrm{Im}\,\lambda} \left|F(i\sqrt{\lambda})\right|^2 |d\sqrt{\lambda}| = \frac{2\pi}{\varepsilon} \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \left|F(i\sqrt{\lambda})\right|^2 |d\sqrt{\lambda}|. \tag{3.17}$$

Hence we find the estimate

$$\varepsilon \int_{-\infty}^{+\infty} \left\| \frac{y_-(f,\lambda)}{\sqrt{\lambda}} \cdot e^{-\sqrt{-\lambda}x}\chi_+(x)\right\|_{L^2}^2 d\mu \le 2\pi \int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \left|F(i\sqrt{\lambda})\right|^2 |d\sqrt{\lambda}|. \tag{3.18}$$

Finally, let us remark that $|d\sqrt{\lambda}|$ is the Carleson measure for all $\varepsilon > 0$ (see [17]). It is easy to see that the Carleson norms of these measures are uniformly bounded. Then, by the Carleson embedding theorem (see [17]), there exists $C_1 > 0$ such that for all $\varepsilon > 0$ the inequality

$$\int_{i\varepsilon-\infty}^{i\varepsilon+\infty} \left|F(i\sqrt{\lambda})\right|^2 |d\sqrt{\lambda}| \le\ C_1\,\|F\|_{H^2}^2 = C_1\|f_-\|_{L^2} \le C_1\|f\|_{L^2} \tag{3.19}$$

holds. Combining (3.18) and (3.19), one gets (3.14).

Other inequalities can be obtained analogously. □

3.3. Proof of Theorem 3.1

3.3.1. Necessity. Follows immediately from Lemmas 3.2 and 3.3.

3.3.2. Sufficiency. a) Suppose that conditions (i) and (ii) of Theorem 3.1 hold. Suppose also that $\lambda_1 \in \mathbb{C}_+$ is a unique zero of $\varphi_B^+(\cdot)$ and $\varphi_{B^*}^+(\cdot)$ does not vanish in $\mathbb{C}_+$. Then, by Lemma 2.1, $\sigma(A_B) = \mathbb{R} \cup \{\lambda_1\}$.

Denote by $B(\lambda_1)$ a closed neighborhood of λ_1 such that $B(\lambda_1) \subset \mathbb{C}_+$. Let us consider the Riesz projection

$$P_1 := \frac{1}{2\pi i} \int_{\partial B(\lambda_1)} (A_B - \lambda)^{-1} d\lambda \ , \tag{3.20}$$

where $\partial B(\lambda_1)$ is the boundary of $B(\lambda_1)$.

Then (see [25]) $P_1 \in [L^2(\mathbb{R})]$ and $A_B P_1 = P_1 A_B$. Since A_0 is a self-adjoint operator, it follows that

$$\frac{1}{2\pi i} \int_{\partial B(\lambda_1)} \left((A_B - \lambda)^{-1} - (A_0 - \lambda)^{-1} \right) d\lambda = \frac{1}{2\pi i} \int_{\partial B(\lambda_1)} (A_B - \lambda)^{-1} d\lambda. \tag{3.21}$$

It is not hard to show that P_1 is a one-dimensional operator in $L^2(\mathbb{R})$. Actually, we set $m_{\lambda_1} := \lim\limits_{\lambda \to \lambda_1} \frac{\lambda - \lambda_1}{\varphi(\lambda)}$. Using (3.21) and (2.19), one gets

$$\begin{aligned} \frac{(P_1 f)(x)}{m_{\lambda_1}} &= e^{-\sqrt{-\lambda_1} x} \cdot \chi_+(x) \cdot \left(\left(b_{22} + \frac{1}{\sqrt{\lambda_1}} \right) y_+(f, \lambda_1) - \frac{b_{12}}{\sqrt{\lambda_1}} \cdot y_-(f, \lambda_1) \right) + \\ &+ \frac{e^{\sqrt{\lambda_1} x} \cdot \chi_-(x)}{\sqrt{\lambda_1}} \left(\frac{b_{11} + \sqrt{-\lambda_1}}{\sqrt{\lambda_1}} \cdot y_-(f, \lambda_1) - b_{21} \cdot y_+(f, \lambda_1) \right), \qquad f \in L^2(\mathbb{R}). \end{aligned} \tag{3.22}$$

Let us write (3.22) in the following form

$$\begin{aligned} \frac{(P_1 f)(x)}{m_{\lambda_1}} &= y_+(f, \lambda_1) \cdot \left[e^{-\sqrt{-\lambda_1} x} \chi_+(x) \cdot \left(b_{22} + \frac{1}{\sqrt{\lambda_1}} \right) - b_{21} \frac{e^{\sqrt{\lambda_1} x} \cdot \chi_-(x)}{\sqrt{\lambda_1}} \right] \\ &+ \frac{y_-(f, \lambda_1)}{\sqrt{\lambda_1}} \cdot \left[e^{-\sqrt{-\lambda_1} x} \chi_+(x) \cdot (-b_{12}) + \frac{e^{\sqrt{\lambda_1} x} \cdot \chi_-(x)}{\sqrt{\lambda_1}} \cdot \left(b_{11} + \sqrt{-\lambda_1} \right) \right] . \end{aligned} \tag{3.23}$$

Note that

$$\det \begin{pmatrix} (b_{22} + \frac{1}{\sqrt{\lambda_1}}) & -b_{12} \\ -b_{21} & (b_{11} - i\sqrt{\lambda_1}) \end{pmatrix} = \det(B - M(\lambda_1)) = \varphi_B(\lambda_1) = 0. \tag{3.24}$$

Hence P_1 is a one-dimensional operator.

b) By step (a), the space $\mathfrak{H} = L^2(\mathbb{R})$ can be decomposed as (see [25])

$$\mathfrak{H} = \mathfrak{H}_0 \dotplus \mathfrak{H}_1, \qquad \mathfrak{H}_j := P_j \mathfrak{H}, \quad j \in \{0,1\}, \qquad P_0 := I - P_1. \tag{3.25}$$

Moreover, $A_B P_j = P_j A_B, j \in \{0,1\}$, and the operator A_B admits the following decomposition

$$A_B = A_B^0 \dotplus A_B^1, \qquad A_B^j := P_j A_B P_j, \quad j \in \{0,1\}. \tag{3.26}$$

We also have $\sigma(A_B^1) = \{\lambda_1\}$ and $\sigma(A_B^0) = \mathbb{R}$.

Let us show that the inequality

$$\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \varepsilon \left\| R_{A_B^0} (\mu + i\varepsilon) f \right\|^2 d\mu \le C \|f\|^2, \qquad f \in \mathfrak{H}_0, \tag{3.27}$$

holds with some constant $C > 0$.

Since $\varphi_B^+(\cdot)$ does not vanish in $\mathbb{R} \cup \{\infty\}$, we see that Lemma 3.4 and (2.19) imply

$$\sup_{\varepsilon>0} \int_{\substack{\mu\in\mathbb{R} \\ \mu+i\varepsilon\notin B(\lambda_1)}} \|R_{A_B}(\mu + i\varepsilon)f\|^2 d\mu \le C_1\|f\|^2, \qquad f \in L^2(\mathbb{R}) \ . \tag{3.28}$$

Therefore

$$\sup_{\varepsilon>0} \int_{\substack{\mu\in\mathbb{R} \\ \mu+i\varepsilon\notin B(\lambda_1)}} \|R_{A_B^0}(\mu + i\varepsilon)f\|^2 d\mu \le C_1\|f\|^2, \qquad f \in \mathfrak{H}_0. \tag{3.29}$$

Further, let us recall that $\lambda_1 \in \rho(A_B^0)$. It means that the operator-function $R_{A_B^0}(\lambda)$ is bounded on $B(\lambda_1)$. If we combine this with (3.29), we get (3.27).

Since $\sigma(A_B^0) = \mathbb{R}$ and $\varphi_{B^*}^+(\cdot)$ have no zeroes in $\overline{\mathbb{C}_+}$, we can obtain in the same way the estimate

$$\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \|R_{(A_B^0)^*}(\mu + i\varepsilon)f\|^2 d\mu \le C^*\|f\|^2, \qquad f \in \mathfrak{H}_0, \qquad C^* = const > 0. \tag{3.30}$$

Hence, by Theorem 1.2, the operator A_B^0 is similar to a self-adjoint operator. Moreover, A_B^1 is a one-dimensional operator. Thus the operator A_B is similar to a normal one.

c) General case. Suppose that conditions (i) and (ii) hold, i.e., $\varphi_B^+(\cdot)$ and $\varphi_{B^*}^+(\cdot)$ do not vanish on $\mathbb{R} \cup \{\infty\}$ and have only simple zeros in $\mathbb{C}_+$. Let us denote by $n(\varphi_B^+)$ the number of zeroes of $\varphi_B^+(\cdot)$ in $\mathbb{C}_+$. Then $n(A_B) := n(\varphi_B^+) + n(\varphi_{B^*}^+)$ is the number of eigenvalues of A_B, $\sigma_p(A_B) = \{\lambda_i : i = 1, \ldots, n(A_B)\}$, ($\sigma_p(A_B) = \emptyset$ if $n(A_B) = 0$).

By (3.4), the function $\varphi_B^+(\cdot)$ has at most two zeroes in $\mathbb{C}_+$. Hence $n(A_B) \le 4$. It can be shown in the same way as in step (b) that exists a decomposition

$$L^2(\mathbb{R}) = \mathfrak{H}_0 \dotplus \mathfrak{H}_1 \dotplus ... \dotplus \mathfrak{H}_{n(A_B)},$$

$$A_B = A_B^0 \dotplus A_B^1 \dotplus ... \dotplus A_B^{n(A_B)}. \tag{3.31}$$

Here A_B^0 is similar to a self–adjoint operator and A_B^i are one-dimensional operators.

Thus, A_B is similar to a normal operator. This completes the proof.

4. Other boundary conditions

4.1. The case $\Delta_2 \neq 0$

The following theorem is an obvious corollary of Theorem 2.1.

Theorem 4.1. *(i) The triplet* $\Pi_2 = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$*, where* $\Gamma_j : W_2^2(\mathbb{R}\setminus\{0\}) \to \mathbb{C}^2, j \in \{0,1\}$,

$$\Gamma_0 f = \begin{pmatrix} -f'(+0) \\ f(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f(+0) \\ f'(-0) \end{pmatrix}, \tag{4.1}$$

is a boundary triplet for A^*.

(ii) The corresponding Weyl function $M(\cdot)$ *is*

$$M(\lambda) = M_2(\lambda) := \begin{pmatrix} 1/\sqrt{-\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{4.2}$$

(iii) The corresponding γ*–field* $\gamma(\lambda) : \mathbb{C}^2 \to \mathfrak{N}_\lambda$ *is*

$$\gamma(\lambda)\begin{pmatrix} c_+ \\ c_- \end{pmatrix} := c_+ \frac{1}{\sqrt{-\lambda}} \cdot e^{-\sqrt{-\lambda}x}\chi_+(x) + c_- \cdot e^{\sqrt{\lambda}x}\chi_-(x), \qquad c_\pm \in \mathbb{C}. \tag{4.3}$$

If $\Delta_2 \neq 0$ then the boundary conditions (2.8) take the form

$$\begin{cases} f(+0) = -b_{11}f'(+0) + b_{12}f(-0) \\ f'(-0) = -b_{21}f'(+0) + b_{22}f(-0) \end{cases}. \tag{4.4}$$

Further, by Definition 1.5, we get

$$A_{(a_{ij})} = A_B = A^*|\ker(\Gamma_1 - B\Gamma_0), \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}. \tag{4.5}$$

As before, denote by $\varphi_B(\cdot)$ the function as in (2.14) with $M(\cdot)$ as in (4.2).

Combining Proposition 1.2 and Theorem 1.2, one obtains

Lemma 4.1. *If* A_B *is the operator* (4.5) *and* $|b_{12}| + |b_{21}| \neq 0$ *then:*

(i) $\sigma_c(A_B) = \mathbb{R}$, $\quad \sigma_r(A_B) = \emptyset$;

(ii) $\sigma_p(A_B) = \{\lambda \in \mathbb{C}\setminus\mathbb{R} : \varphi_B(\lambda) = 0\} = \{\lambda \in \mathbb{C}_+ : \varphi_B(\lambda) = 0\} \cup \{\lambda \in \mathbb{C}_- : \varphi_{B^*}(\overline{\lambda}) = 0\}$.

(iii) The Krein formula has the form

$$\begin{aligned} &\left((A_B - \lambda)^{-1} f - (A_0 - \lambda)^{-1} f\right)(x) = \\ &\quad = \frac{e^{-\sqrt{-\lambda}x} \cdot \chi_+(x)}{\varphi_B(\lambda)\cdot\sqrt{-\lambda}} \left(\frac{b_{22} - \sqrt{\lambda}}{\sqrt{-\lambda}} \cdot y_+(f,\lambda) - b_{12}\cdot y_-(f,\lambda)\right) + \\ &\quad + \frac{e^{\sqrt{\lambda}x}\cdot\chi_-(x)}{\varphi_B(\lambda)} \left(-\frac{b_{21}}{\sqrt{-\lambda}} \cdot y_+(f,\lambda) + \left(b_{11} - \frac{1}{\sqrt{-\lambda}}\right)\cdot y_-(f,\lambda)\right), \\ &\qquad f \in L^2(\mathbb{R}), \end{aligned} \tag{4.6}$$

where $A_0 = A^*|\ker\Gamma_0$, $\lambda \in \rho(A_B)\cap\rho(A_0)$*, and* $y_\pm(f,\lambda)$ *are defined by* (2.18).

4.2. The case $\Delta_3 \neq 0$

Let $\Delta_3 \neq 0$. In this case we write the boundary conditions (2.8) in the following form

$$\begin{cases} f'(+0) = b_{11}f(+0) + b_{12}f(-0) \\ f'(-0) = b_{21}f(+0) + b_{22}f(-0). \end{cases} \tag{4.7}$$

Let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and

$$A_B := A^*\lceil \operatorname{dom}(A_B), \qquad \operatorname{dom}(A_B) = \{f \in W_2^2(\mathbb{R}\setminus\{0\}) : f \text{ satisfies (4.7)}\}. \tag{4.8}$$

Theorem 4.2. *(i) The triplet* $\Pi_3 = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, *where* $\Gamma_j : W_2^2(\mathbb{R}\setminus\{0\}) \to \mathbb{C}^2, j \in \{0,1\}$,

$$\Gamma_0 f = \begin{pmatrix} f(+0) \\ f(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f'(+0) \\ f'(-0) \end{pmatrix}, \tag{4.9}$$

is a boundary triplet for A^*.

(ii) The corresponding Weyl function $M(\cdot)$ *is*

$$M(\lambda) = M_3(\lambda) := \begin{pmatrix} -\sqrt{-\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{4.10}$$

(iii) The corresponding γ*–field* $\gamma(\lambda) : \mathbb{C}^2 \to \mathfrak{N}_\lambda$ *is*

$$\gamma(\lambda)\begin{pmatrix} c_+ \\ c_- \end{pmatrix} := c_+ \cdot e^{-\sqrt{-\lambda}x}\chi_+(x) + c_- \cdot e^{\sqrt{\lambda}x}\chi_-(x), \qquad c_\pm \in \mathbb{C}. \tag{4.11}$$

Therefore, A_B of the form (4.8) is an almost solvable extension of A and $A_B = A^*\lceil \ker(\Gamma_1 - B\Gamma_0)$, where Γ_i, $i \in \{0,1\}$, are defined by (4.9).

Lemma 4.2. *Let the function* $\varphi_B(\cdot)$ *be of the form* (2.14) *with* $M(\cdot)$ *defined by* (4.10)*; let the operator* A_B *is given by* (4.8) *and* $|b_{12}| + |b_{21}| \neq 0$*. Then:*

(i) $\sigma_c(A_B) = \mathbb{R}, \quad \sigma_r(A_B) = \emptyset$*;*

(ii) $\sigma_p(A_B) = \{\lambda \in \mathbb{C}\setminus\mathbb{R} : \varphi_B(\lambda) = 0\} = \{\lambda \in \mathbb{C}_+ : \varphi_B(\lambda) = 0\} \cup \{\lambda \in \mathbb{C}_- : \varphi_{B^*}(\overline{\lambda}) = 0\}$.

(iii) The Krein formula has the form

$$\begin{aligned} &\left((A_B - \lambda)^{-1}f - (A_0 - \lambda)^{-1}f\right)(x) = \\ &\qquad = \frac{e^{-\sqrt{-\lambda}x} \cdot \chi_+(x)}{\varphi_B(\lambda)}\left((b_{22} - \sqrt{\lambda}) \cdot y_+(f,\lambda) - b_{12} \cdot y_-(f,\lambda)\right) + \\ &+ \frac{e^{\sqrt{\lambda}x} \cdot \chi_-(x)}{\varphi_B(\lambda)}\left(-b_{21} \cdot y_+(f,\lambda) + (b_{11} + \sqrt{-\lambda}) \cdot y_-(f,\lambda)\right), \qquad f \in L^2(\mathbb{R}), \end{aligned} \tag{4.12}$$

where $A_0 = A^*\lceil \ker \Gamma_0$, $\lambda \in \rho(A_B) \cap \rho(A_0)$, *and* $y_\pm(f,\lambda)$ *are given by* (2.18).

4.3. The case $\Delta_4 \neq 0$

If $\Delta_4 \neq 0$, then boundary conditions (2.8) take the form

$$\begin{cases} -f(+0) = b_{11}f'(+0) + b_{12}f'(-0) \\ -f(-0) = b_{21}f'(+0) + b_{22}f'(-0). \end{cases} \tag{4.13}$$

For $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ we set

$$A_B := A^* \upharpoonright \operatorname{dom}(A_B),$$
$$\operatorname{dom}(A_B) = \{f \in W_2^2(\mathbb{R} \setminus \{0\}) : f \text{ satisfies (4.13)}\}. \tag{4.14}$$

Theorem 4.3. *(i) The triplet $\Pi_4 = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where $\Gamma_j : W_2^2(\mathbb{R} \setminus \{0\}) \to \mathbb{C}^2, j \in \{0,1\}$,*

$$\Gamma_0 f = \begin{pmatrix} f'(+0) \\ f'(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} -f(+0) \\ -f(-0) \end{pmatrix}, \tag{4.15}$$

is a boundary triplet for A^*.

(ii) The corresponding Weyl function $M(\cdot)$ is

$$M(\lambda) = M_4(\lambda) := \begin{pmatrix} 1/\sqrt{-\lambda} & 0 \\ 0 & -1/\sqrt{\lambda} \end{pmatrix}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.16}$$

(iii) The corresponding γ-field $\gamma(\lambda) : \mathbb{C}^2 \to \mathfrak{N}_\lambda$ is

$$\gamma(\lambda) \begin{pmatrix} c_+ \\ c_- \end{pmatrix} := -\frac{c_+}{\sqrt{-\lambda}} \cdot e^{-\sqrt{-\lambda}x} \chi_+(x) + \frac{c_-}{\sqrt{\lambda}} \cdot e^{\sqrt{\lambda}x} \chi_-(x), \qquad c_\pm \in \mathbb{C}. \tag{4.17}$$

Therefore, A_B as in (4.14) is an almost solvable extension of A and $A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$, where Γ_j, $j \in \{0,1\}$ are given by (4.15).

Lemma 4.3. *Suppose the function $\varphi_B(\cdot)$ is given by (2.14) with $M(\cdot)$ as in (4.16). If the operator A_B is given by (4.14) and $|b_{12}| + |b_{21}| \neq 0$ then:*

(i) $\sigma_c(A_B) = \mathbb{R}$, $\quad \sigma_r(A_B) = \emptyset$;

(ii) $\sigma_p(A_B) = \{\lambda \in \mathbb{C} \setminus \mathbb{R} : \varphi_B(\lambda) = 0\} = \{\lambda \in \mathbb{C}_+ : \varphi_B(\lambda) = 0\} \cup \{\lambda \in \mathbb{C}_- : \varphi_{B^*}(\overline{\lambda}) = 0\}$.

(iii) The Krein formula has the form

$$\begin{aligned} &\left((A_B - \lambda)^{-1} f - (A_0 - \lambda)^{-1} f\right)(x) = \\ &\quad = \frac{e^{-\sqrt{-\lambda}x} \cdot \chi_+(x)}{\varphi_B(\lambda) \cdot \sqrt{-\lambda}} \left((b_{22} + 1/\sqrt{\lambda}) \cdot \frac{y_+(f,\lambda)}{\sqrt{-\lambda}} + b_{12} \cdot \frac{y_-(f,\lambda)}{\sqrt{\lambda}} \right) + \\ &+ \frac{e^{\sqrt{\lambda}x} \cdot \chi_-(x)}{\varphi_B(\lambda) \cdot \sqrt{\lambda}} \left(b_{21} \cdot \frac{y_+(f,\lambda)}{\sqrt{-\lambda}} + (b_{11} - 1/\sqrt{-\lambda}) \cdot \frac{y_-(f,\lambda)}{\sqrt{\lambda}} \right), \qquad f \in L^2(\mathbb{R}), \end{aligned} \tag{4.18}$$

where $A_0 = A^* \upharpoonright \ker \Gamma_0$, $\lambda \in \rho(A_B) \cap \rho(A_0)$, and $y_\pm(f,\lambda)$ are given by (2.18).

4.4. The similarity criterion

Arguing as above, we see that for the cases considered in subsections 4.1-4.3 analogues of Theorems 3.1 and 3.2 hold.

Theorem 4.4. *Theorems* 3.1 *and* 3.2 *are valid for the extensions* A_B *with boundary conditions* (4.4), (4.7) *or* (4.13) *if the function* $M(\cdot)$ *is replaced by* (4.2), (4.10) *or* (4.16), *respectively.*

5. On similarity of $(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + c\delta\right)$ and $(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + c\delta'\right)$ to normal operators

5.1. Let us illustrate the previous results by several examples. We start with the operator $\tilde{A} = -(\operatorname{sgn} x)\frac{d^2}{dx^2}$, see (0.3). It is obvious that $\tilde{A} = A_B$, where A_B is an almost solvable extension of the form (2.13) with $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It follows from (3.4) that in this case

$$\varphi_B^+(\lambda) = \varphi_{B^*}^+(\lambda) \equiv 1 - i, \qquad \lambda \in \overline{\mathbb{C}_+}.$$

Hence, by Theorem 3.2, one obtains the result of [7].

Theorem 5.1 ([7])**.** *The operator* $\tilde{A} = -(\operatorname{sgn} x)\frac{d^2}{dx^2}$ *is similar to a self–adjoint operator.*

5.2. Let δ be the Dirac delta. Let us introduce the following differential expression

$$-\frac{d^2}{dx^2} + c\delta, \qquad c \in \mathbb{C} \setminus \{0\}. \tag{5.1}$$

Here δ formally represents a contact interaction at zero if $c \in \mathbb{R}$.

In $L^2(\mathbb{R})$, expression (5.1) generates the differential operator $L_{c\delta}$ defined by

$$\operatorname{dom}(L_{c\delta}) = \{f \in W_2^2(\mathbb{R} \setminus \{0\}) : f(+0) = f(-0), f'(+0) - f'(-0) = cf(-0)\},$$

$$L_{c\delta} := -\frac{d^2}{dx^2} \tag{5.2}$$

(see for example [2, 35]). We put $A_{c\delta} := JL_{c\delta}$, where $(Jf)(x) = (\operatorname{sgn} x)f(x)$, i.e., the operator $A_{c\delta}$ is defined in $L^2(\mathbb{R})$ by the differential expression

$$(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + c\delta\right), \qquad c \in \mathbb{C} \setminus \{0\}. \tag{5.3}$$

It is clear that the operator $A_{c\delta}$ is an extension of the operator A of the form (0.1). Moreover, $A_{c\delta} = A_B$ where A_B is given by (2.13) and

$$B = B_c := \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.4}$$

Theorem 5.2. *Let $c \neq 0$.*

(i) The operator $A_{c\delta}$ is similar to a normal one if and only if $\operatorname{Re} c \neq -|\operatorname{Im} c|$.

(ii) $A_{c\delta}$ is similar to a self–adjoint operator if and only if $\operatorname{Re} c > -|\operatorname{Im} c|$.

Proof. Let B be given by (5.4). By (3.4), one gets

$$\varphi_B^+(\lambda) = 1 - i + \frac{c}{\sqrt{\lambda}}, \qquad \varphi_{B^*}^+(\lambda) = 1 - i + \frac{\overline{c}}{\sqrt{\lambda}}, \qquad \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}. \tag{5.5}$$

Note that $\varphi_B^+(\cdot)$ or $\varphi_{B^*}^+(\cdot)$ have a real zero iff $\operatorname{Re}(c) = -|\operatorname{Im}(c)|$. Furthermore, $\varphi_B^+(\cdot)$ and $\varphi_{B^*}^+(\cdot)$ do not vanish in $\overline{\mathbb{C}_+}$ iff $\operatorname{Re}(c) > -|\operatorname{Im} c|$. Hence the statements of Theorem 5.2 obviously follow from Theorems 3.1 and 3.2. □

5.3. Let us consider the extension $A_{\widetilde{B}}$ of the form (2.13) with

$$\widetilde{B} = \widetilde{B}_c = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix}, \ c \neq 0.$$

This is a so-called "operator with δ'–interaction" (see [2]). The formal differential expression corresponding to $A_{\widetilde{B}}$ is

$$(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + c\delta'\right), \qquad c \in \mathbb{C} \setminus \{0\}. \tag{5.6}$$

Therefore we will denote the operator $A_{\widetilde{B}}$ by $A_{c\delta'}$.

Theorem 5.3. *Let $c \neq 0$.*

(i) The operator $A_{c\delta'}$ is similar to a normal one if and only if $\operatorname{Re} c \neq -|\operatorname{Im} c|$.

(ii) $A_{c\delta'}$ is similar to a self–adjoint operator if and only if $\operatorname{Re} c > -|\operatorname{Im} c|$.

Proof. So, by (3.4), we have

$$\varphi_{c\delta'}^+(\lambda) = 1 - i - ic\sqrt{\lambda}, \qquad \varphi_{c\delta'}^+(\lambda) = 1 - i - i\overline{c}\sqrt{\lambda}, \qquad \lambda \in \overline{\mathbb{C}_+}. \tag{5.7}$$

These functions have a real zero iff $\operatorname{Re} c = -|\operatorname{Im} c|$, have no zeros in $\mathbb{C}_+$ iff $\operatorname{Re} c > -|\operatorname{Im} c|$. Hence the statements of Theorem 5.3 follow from Theorems 3.1 and 3.2. □

Remark 5.1. *Let $\widehat{L}$ be a self-adjoint extension of the symmetric operator*

$$(Lf)(x) := -f''(x), \quad \operatorname{dom}(L) = \{f \in W_2^2(\mathbb{R}) : f(0) = f'(0) = 0\}. \tag{5.8}$$

Assume that the boundary conditions at 0 associated with the extension $\widehat{L}$ are nonseparate, i.e., the operator $\widehat{L}$ does not admit the following decomposition $\widehat{L}_+ \oplus \widehat{L}_-$, where $\widehat{L}_\pm := \widehat{L}\,|(\operatorname{dom}(\widehat{L}) \cap L^2(\mathbb{R}_\pm))$. Then $\widehat{L}$ can be considered as an operator with a singular interaction (see [27] for the details). Using the arguments of this section, one can describe the main spectral properties of the corresponding J-self-adjoint operator $\widehat{A} := J\widehat{L}$.

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